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THE SPIN L-FUNCTION ON THE SYMPLECTIC GROUP $\mathbf{GSp}(6)$

BY

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ABSTRACT

In this paper, we will study some essential analytic properties of the "spin" L-function on the symplectic group $\mathbf{GSp}(6)$ (which is associated with the eight-dimensional spin representation of the L-group $\mathrm{Gspin}(7,\mathbb{C})$), namely, uniqueness of a bilinear form on an irreducible admissible representation of $\mathbf{GSp}(6) \times \mathbf{GL}(2)$, local functional equation, and meromorphic continuation, non-vanishing properties at non-archimedean places as well as at archimedean places.

Consequently, we will determine the location of the possible poles of the global spin L-function of a generic automorphic cuspidal representation of $\mathbf{GSp}(6)$.

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0. Introduction

We will prove here some essential analytic properties of the spin L-functions on the symplectic group $\mathbf{GSp}(6)$, which is associated with the eight-dimensional spin representation of the L-group $\text{Gspin}(7, \mathbb{C})$.

At non-archimedean places, we will prove propositions about uniqueness of a bilinear form on an irreducible admissible representation of $\mathbf{GSp}(6) \times \mathbf{GL}(2)$, local functional equation, meromorphic continuation and non-vanishing property.

At archimedean places, we will also prove some analogous propositions about meromorphic continuation and a non-vanishing property.

Consequently, we will determine the location of the possible poles of the global spin L-function of a generic automorphic cuspidal representation π of $\mathbf{GSp}(6)$.

We define this L-function as an Euler product:

$$L_S(s, \pi, \text{spin}) = \prod_{v \notin S} L_v(s, \pi_v, \text{spin}),$$

where S is a finite set of places (including archimedean places) such that each place $v \notin S$ is unramified.

For each place $v \notin S$, let O_v be its ring of integers. This ring has a unique maximal ideal $\mathbf{p}_v$. Then the cardinality of the residue field is $q_v = |O_v/\mathbf{p}_v|$.

The connected L-group ${}^LG^o$ of $\mathbf{GSp}(6, \mathbf{F})(6)$ is $\text{Gspin}(7, \mathbb{C})$ which has an irreducible 2^3-dimensional spin representation r_v: $\text{Gspin}(7, \mathbb{C}) \longrightarrow \text{GL}(8, \mathbb{C})$.

By the Satake isomorphism, there is a bijection between π_v and ${}^LG^o$-semisimple conjugacy class t_v in ${}^LG_v$. Then $r_v(t_v)$ has eight eigenvalues of the form $\alpha_{v,1}^{\pm 1}\alpha_{v,2}^{\pm 1}\alpha_{v,3}^{\pm 1}$. Then the local L-function (where $v \notin S$) is defined by

$$L_v(s, \pi_v, \text{spin}) = \prod_{\text{eight factors}} (1 - \alpha_{v,1}^{\pm 1}\alpha_{v,2}^{\pm 1}\alpha_{v,3}^{\pm 1} . q_v^{-s})^{-1} = \det[I_8 - r_v(t_v)q_v^{-s}]^{-1},$$

where I_8 is the 8×8 identity matrix.

In the first chapter, we will deal with non-archimedean places. Let $\mathbf{F}$ be a non-archimedean local field, let π: $\mathbf{GSp}(6, \mathbf{F}) \longrightarrow \text{End}(V_\pi)$ an irreducible, smooth and generic cuspidal representation of $\mathbf{GSp}(6,\mathbf{F})$ and let $\rho\colon \mathbf{GL}(2,\mathbf{F}) \longrightarrow \mathrm{End}(\mathbf{W}_\rho)$ be the representation $\mathrm{ind}_{\mathbf{B}_2}^{\mathbf{GL}(2,\mathbf{F})}(\delta_{\mathbf{B}_2}^s)$ obtained by nonnormalized induction, where $\delta_{\mathbf{B}_2}$ is a modular character of the Borel subgroup $\mathbf{B}_2$ of $\mathbf{GL}(2,\mathbf{F})$.

Theorem 1 will show *the uniqueness of a bilinear form* $B(v,w)$ *on a representation of* $\mathbf{GSp}(6,\mathbf{F}) \times \mathbf{GL}(2,\mathbf{F})$ *which satisfies*

$$B\left[\pi\begin{pmatrix} I & X' & Y \\ & I & X \\ & & I\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & g\end{pmatrix} v, \rho(g)w\right] = \psi_o(\mathrm{trace}(X)).B(v,w),$$

for all g *in* $\mathbf{GL}(2,\mathbf{F})$, $v \in V_\pi$ *and* $w \in W_\rho$, *where* ψ_o *is some fixed non-trivial unitary additive character of* $\mathbf{F}$, *and the matrices* I, X, X', Y *are* 2×2 *matrices whose entries are in* $\mathbf{F}$ *and* I *is the identity matrix.*

Now let

$$\mathbf{U}_1 = \left\{u_1 \in \mathbf{GSp}(6,\mathbf{F}) \mid u_1 = \begin{pmatrix} I & X' & Y \\ & I & X \\ & & I\end{pmatrix}\begin{pmatrix} b_1 & & \\ & b_1 & \\ & & b_1\end{pmatrix}\right\},$$

where

$$b_1 \in \mathbf{B}_2' = \left\{\begin{pmatrix} * & * \\ & 1\end{pmatrix} \in \mathbf{GL}(2,\mathbf{F})\right\}.$$

Let $\psi(X) = \psi_o(\mathrm{trace}(X))$ and χ be the restriction on $\mathbf{B}_2'$ of the modular character $\chi_o = \delta_{\mathbf{B}_2}^s$. of the Borel subgroup $\mathbf{B}_2$.

Let $\mathbf{P}_6$ be a subgroup consisting of matrices of the form

$$\begin{pmatrix} * & * & * \\ & \mathbf{GSp}(4,\mathbf{F}) & * \\ & & 1\end{pmatrix}$$

in $\mathbf{P}^{1,2}$ which is the standard maximal parabolic subgroup of the symplectic group $\mathbf{GSp}(6,\mathbf{F})$ whose Levi factor is isomorphic to $\mathbf{GL}(1,\mathbf{F}) \times \mathbf{GSp}(4,\mathbf{F})$. These subgroups will be described explicitly in the next section.

Thus the problem can be reduced to proving that:

For π *an irreducible, smooth and generic representation of* $\mathbf{GSp}(6,\mathbf{F})$, *the dimension of the space* $\mathrm{Hom}_{\mathbf{P}_6}(\pi_{\mathbf{P}_6}, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))$ *is at most one.*

The essential tools used are Jacquet functors and their adjoint functors. We will state and prove some important properties of these functors which are similar to those in proposition 5.12 in the paper of J. Berstein and A. Zelevinsky [B,Z].

These functors help to reduce from working on the subgroup $\mathbf{P}_6$, through several stages, to working on the symplectic subgroups of lower dimension and their parabolic subgroups.

The analysis here is unusual because we will do the above descent through a series of parabolic subgroups similar to the "derivatives" of Berstein-Zelevinsky which are emphasized in [G,PS] but, in contrast with their work, we will use parabolic subgroups with non-abelian subgroups. For this reason, each descent will be performed in two stages, except the final descent, as shown in the following chart.

The chart could give readers a clear view of stages of the double coset calculations which will be done in Sections 7 and 8. The notations in this chart may not be standard but will not cause any confusion. Please also refer to Section 3.

$$\mathbf{P}_6 \leftrightarrow \pi$$

$$\swarrow \quad \searrow$$

$$\mathcal{J}_-^{\mathbf{L},1} \qquad \mathcal{J}_-^{\mathbf{L},o} \qquad \text{(Stage 1, in Section 7)}$$

$$\swarrow \qquad\qquad \searrow$$

$$\theta_1 \leftrightarrow (\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1}) \qquad (\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o}) \leftrightarrow \theta_o$$

$$\swarrow \quad \searrow$$

$$\mathcal{J}_-^{\mathbf{K},1} \qquad \mathcal{J}_-^{\mathbf{K},o} \qquad \text{(Stage 2, in Section 7)}$$

$$\swarrow \qquad\qquad \searrow$$

$$\sigma_1 \leftrightarrow (\mathbf{P}_4, \Theta_{\mathbf{K},1}) \qquad (\mathbf{G}_4, \Theta_{\mathbf{K},o}) \leftrightarrow \sigma_o$$

$$\swarrow \quad \searrow$$

$$\mathcal{J}_-^{\mathbf{L},1} \qquad \mathcal{J}_-^{\mathbf{L},o} \qquad \text{(Section 8)}$$

$$\swarrow \qquad\qquad \searrow$$

$$\lambda_1 \leftrightarrow (\mathbf{S}_2.\mathbf{T}_4, \Lambda_{\mathbf{L},1}) \qquad (\mathbf{G}_2.\mathbf{T}_4, \Lambda_{\mathbf{L},o}) \leftrightarrow \lambda_o$$

$$\swarrow \quad \searrow$$

$$\mathcal{J}_-^{\mathbf{K},1} \qquad \mathcal{J}_-^{\mathbf{K},o} \qquad \text{(Section 8)}$$

$$\swarrow \qquad\qquad \searrow$$

$$\eta_1 \leftrightarrow (\mathbf{P}_2, \Lambda_{\mathbf{K},1}) \qquad (\mathbf{G}_2, \Lambda_{\mathbf{K},o}) \leftrightarrow \eta_o$$

$$\swarrow \quad \searrow$$

$$\mathcal{J}_-^{1} \qquad \mathcal{J}_-^{o} \qquad \text{(Section 8)}$$

$$\swarrow \qquad\qquad \searrow$$

$$\xi_1 \leftrightarrow (\mathbf{P}_o, \Omega_1) \qquad (\mathbf{G}_o, \Omega_o) \leftrightarrow \xi_o$$

The reduction in each stage is established by the help of an essential lemma which could be considered a version of the Mackey's theorem for the local fields.

This allows us to work in the Hom spaces on the set of double cosets instead of the Hom spaces of the original subgroups.

The double cosets calculations show explicitly that there is at most one double coset on which the Hom space is nontrivial. That is, the Hom space on the lower-dimension subgroup will be carried to the next stage and so on. We then have at the final stage the Whittaker model on $\mathbf{GL}(2, \mathbf{F})$ which proves our problem.

One immediate result of Theorem 1 is the local functional equation for the p-adic field. Let us recall the definition of the integral $Z(s, W, f_s)$ in [B,G]:

$$Z(s, W, f_s) = \prod_v Z_v(s, W_v, f_{s,v}),$$

where

$$\begin{aligned} Z_v(s, W_v, f_{s,v}) =& \zeta_{\mathbf{F}_v}(2s) \\ &\times \int_{\mathbf{B}_{2,v}\backslash \mathrm{GL}\,2v} \int_{\mathbf{F}_v^\times} \int_{\mathbf{F}_v^2} W_v\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix} .\gamma^{-1}.\mathrm{i}(g_v)\right) \\ &\times |a|^{s-3}.f_{s,v}(g_v)\, dz\, du\, d^\times a\, dg_v, \end{aligned}$$

and

$$\gamma = \begin{pmatrix} 1 &&&&& \\ &&& -1 && \\ & -1 &&&& \\ &&&& 1 & \\ && -1 &&& \\ &&&&& 1 \end{pmatrix}, \quad \mathrm{i}(g_v) = \begin{pmatrix} g_v && \\ & g_v & \\ && g_v \end{pmatrix}, \quad g_v \in \mathrm{GL}(2, F_v).$$

An identity from Theorem 1 in [B,G] showed that, for almost all places $v \notin S$,

$$Z_v(s, W_v, f_{s,v}) = L_v(s, \pi_v, \mathrm{spin}).$$

PROPOSITION 10.2 (Meromorphic Continuation): *Let $\mathbf{F}_v$ be a non-archimedean local field whose residue field is of cardinality q. Then the integral $Z_v(s, W_v, f_{s,v})$ defines a rational function of variable q^{-s}. Hence, particularly, $Z_v(s, W_v, f_{s,v})$ has a meromorphic continuation to all s.*

The proof is similar to a result of S. Gelbart and I. Piatetski-Shapiro in [G,PS]. They use Bernstein's theorem about analytic continuation of local integrals (proved in his letter [Be] to Piatetski-Shapiro in Fall 1985).

To establish the local functional equation, we will need to prove another property.

PROPOSITION 10.4 (Non-vanishing property): *For any non-archimedean local place v, there exist a Whittaker function $W_v^o \in \mathcal{W}_{\pi_v}$ and a function $f_{s,v}^o \in \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s$ such that $\check{Z}_v(s, W_v^o, f_{s,v}^o) \equiv 1$.*

The proof is rather simple. First, we choose the Whittaker function such that the integral does not vanish, by matrix manipulation and using Schwartz functions. Then the smooth function $f_{s,v}^o \in \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s$ is a locally constant function chosen properly.

We recall the definition of an intertwining operator which will be used in the proof of Proposition 10.3. Let $M_{s,v}$: $\mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathrm{GL}\,2v} \delta_{\mathbf{B}_{2,v}}^s \to \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathrm{GL}\,2v} \delta_{\mathbf{B}_{2,v}}^{1-s}$ be a normalized intertwining operator defined as

$$(M_{s,v}f_{s,v})(g) = \int_{\mathbf{F}_v} f_{s,v}\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot g\right) dx,$$

for all $f_{s,v}$ in V_{ρ_s}. This proves that $V_{\rho_s} \simeq V_{\rho_{1-s}}$ (Jacquet–Langlands' theorem).

This theorem and the above results are sufficient to establish the local functional equation below.

PROPOSITION 10.3 (The local functional equation): *Assume that $\mathbf{F}_v$ is a non-archimedean local field whose residue cardinality is q. Then there exists a meromorphic function $\gamma_v(s)$ such that, for almost all s,*

$$Z_v(s, W_v, f_{s,v}) = \gamma_v(s).Z_v(1 - s, W_v, M_{s,v}f_{s,v}).$$

In fact, $\gamma_v(s)$ is a rational function of q^{-s}.

In the second chapter, we will deal with archimedean places. Then we will be able to determine the locations of the possible poles of the global spin L-function. Indeed, we will need some necessary properties of the integral $Z(s, W_v, f_{s,v})$ at the archimedean places.

First, we will prove the following proposition.

PROPOSITION 11.1 (Meromorphic continuation of $Z_v(s, W_v, f_{s,v})$ at archimedean places): *The integral*

$$Z_v(s,W_v,f_{s,v}) = \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v}\int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v^2} W_v\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.\gamma^{-1}.\mathbf{i}(\tilde{k})\right) \times |a|^{s-3}.f_{s,v}(\tilde{k})\,dz\,du\,d^\times a\,d\tilde{k}$$

converges for sufficiently large $\Re(s)$ and has a meromorphic continuation to all s.

We refer to the work of H. Jacquet, J. Shalika and Piatetski-Shapiro in [J,PS,S], [J,S.1]and [J,S.2], in order to estimate the Whittaker functions. Then the proof of the meromorphic continuation part is reduced to a calculus task.

PROPOSITION 12.1 (Non-vanishing property at archimedean places): *Let v be an archimedean local place. For any s_o fixed, there exist a Whittaker function $W_v^o \in \mathcal{W}_{\pi_v}$ and a $\mathbf{K}_v$-finite function $f_{s,v}^o \in \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}}\delta_{\mathbf{B}_{2,v}}^s$ such that the meromorphic continuation of the integral*

$$Z_v(s,W_v^o,f_{s,v}^o) = \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v}\int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v^2} W_v^o\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.\gamma^{-1}.\mathbf{i}(\tilde{k})\right) \times |a|^{s-3}.f_{s,v}^o(\tilde{k})\,dz\,du\,d^\times a\,d\tilde{k}$$

does not vanish at $s = s_o$.

The proof is the same as that of the non-archimedean case, except in the very final step to choose the smooth function f_v.

In the real place, we will use the Fourier expansion. In the complex case, we will need the Peter–Weyl theorem about matrix coefficients.

All the information about the integral $Z(s,W_v,f_s)$ at all places will help to show the final result of this paper.

THEOREM 13.1: *Let π be an irreducible, smooth and generic cuspidal representation of the symplectic groups $\mathbf{GSp}(6,\mathbf{F})$. The possible poles of the global spin L-function $L_S(s,\pi,spin)$ are only simple poles at $s=0$ and $s=1$.*

In the proof of this theorem, we used results in Theorem 1 in [B,G]:

THEOREM 1 (in [B,G]): *The integral* $Z(s, W, f_s)$ *represents the spin L-function in the sense that for almost all places* v, *the local integral*

$$Z_v(s, W_v, f_{s,v}) = L_v(s, \pi_v, spin).$$

Moreover, it has meromorphic continuation to all s, *with possible poles at* $s = 1$ *and* 0, *and functional equation*

$$Z(s, W, f_s) = Z(1 - s, W, M(s) f_s).$$

Also following from [B,G], we can expect a relationship between the existence of a pole of the L-function and the non-vanishing of a certain global period. That is:

The existence of a pole at $s = 1$ of the global spin L-function $L_S(s, \pi, \text{spin})$ is equivalent to the non-vanishing of the integral

$$\int_{Z_A\, \mathrm{GL}(2,\mathbf{F}) \backslash \mathrm{GL}(2,A)} \int_{(A/\mathbf{F})^7} \left(\begin{pmatrix} 1 & & x_1 & x_2 & x_3 & x_4 \\ & 1 & x_5 & x_6 & x_7 & * \\ & 1 & & * & * & \\ & & & 1 & * & * \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & g \end{pmatrix} \right) dx_1 \cdots dx_7\, dg$$

for some $\phi \in V_\pi$.

Our Theorem 13.1 helped to confirm the above relationship.

A well-known conjecture states that $L_S(s, \pi, \text{spin})$ will have a pole if and only if π is a functorial lift from the exceptional group G_2.

(Please do not confuse with the convenient notation $\mathbf{G}_2$ which is introduced in Section 3.)

CHAPTER I. THE LOCAL FUNCTIONAL EQUATION

1. The Theorem

Let $\mathbf{F}$ be a non-archimedean, local field.

Let $\mathbf{GSp}(6, \mathbf{F}) = \{g \in \mathbf{GL}(6, \mathbf{F}) \mid g.J_6.g^T = \mu(g).J_6$, for some scalar $\mu(g)$ in $\mathbf{F}^\times\}$, where $J_6 = \begin{pmatrix} & & J \\ & J & \\ J & & \end{pmatrix}$ and $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Let $\mathbf{R}$ and $\mathbf{U}$ be two subgroups of $\mathbf{GSp}(6,\mathbf{F})$ which are defined as

$$\mathbf{R} = \left\{ r \in \mathbf{GSp}(6,\mathbf{F}) \mid r = \begin{pmatrix} I & X' & Y \\ & I & X \\ & & I \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & g \end{pmatrix} \right\} \quad \text{and}$$

$$\mathbf{U} = \left\{ u \in \mathbf{GSp}(6,\mathbf{F}) \mid u = \begin{pmatrix} I & X' & Y \\ & I & X \\ & & I \end{pmatrix} \begin{pmatrix} b & & \\ & b & \\ & & b \end{pmatrix} \right\},$$

where $g \in \mathbf{GL}(2,\mathbf{F})$;

$$I, X, X', Y \in \mathrm{Mat}(2,\mathbf{F}) = \{ 2 \times 2 \text{ matrices whose entries are in } \mathbf{F} \},$$

I is unit, and $b \in \mathbf{B}_2 = \left\{ \begin{pmatrix} a & c \\ & d \end{pmatrix} \in \mathbf{GL}(2,\mathbf{F}) \right\}$.

For any fixed s, let us denote $\chi_o = \delta^s_{\mathbf{B}_2}$ where $\delta_{\mathbf{B}_2}$ is a modular character of the Borel subgroup $\mathbf{B}_2$ of $\mathbf{GL}(2,\mathbf{F})$.

We then define an induced representation of $\mathbf{GL}(2,\mathbf{F})$: $\rho = \mathrm{ind}_{\mathbf{B}_2}^{\mathbf{GL}(2,\mathbf{F})}(\chi_o)$, ρ: $\mathbf{GL}(2,\mathbf{F}) \longrightarrow \mathrm{End}(\mathbf{W}_\rho)$, where the induction is non-normalized. Here is the main theorem of this chapter.

THEOREM 1: *Let* π: $\mathbf{GSp}(6,\mathbf{F}) \longrightarrow \mathrm{End}(V_\pi)$ *be an irreducible, smooth and generic cuspidal representation of* $\mathbf{GSp}(6,\mathbf{F})$. *Then, for almost all* s *in the complex plane* $\mathbb{C}$, *there exists at most one bilinear form, up to a constant multiple,* B: $V_\pi \times W_\rho \longrightarrow \mathbb{C}$ *satisfying*

$$B\left[\pi\begin{pmatrix} I & X' & Y \\ & I & X \\ & & I \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & g \end{pmatrix} v, \rho(g)w\right] = \psi_o(\mathrm{trace}(X)).B(v,w), \tag{1.1}$$

for all $v \in V_\pi$ *and* $w \in W_\rho$, *where* ψ_o *is some fixed non-trivial unitary additive character of* $\mathbf{F}$.

We will give the proof of this theorem in Sections 2–9 of this chapter. This theorem will help to establish the p-adic local functional equation in Section 10.

2. The setup

Recalling the definitions of $\mathbf{R}$ and $\mathbf{U}$, we have $r = \begin{pmatrix} I & X' & Y \\ & I & X \\ & & I \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & g \end{pmatrix}$ and $u = \begin{pmatrix} I & X' & Y \\ & I & X \\ & & I \end{pmatrix} \begin{pmatrix} b & & \\ & b & \\ & & b \end{pmatrix}$. Then let $\psi(X) = \psi_o(\mathrm{trace}(X))$.

We can extend the representation ρ of $\mathbf{GL}(2,\mathbf{F})$ to the representation $\rho_{\mathbf{R}}$ by the character ψ on X as: $\rho_{\mathbf{R}}(r)=\psi(X).\rho(g)$.

The character χ_o of $\mathbf{B}_2$ is also extended to the character $\chi_{\mathbf{U}}$ of $\mathbf{U}$ by

$$\chi_{\mathbf{U}}(u)=\psi(X).\chi_o(b).$$

Then

$$\rho_{\mathbf{R}}=\psi.\,\mathrm{ind}_{\mathbf{B}_2}^{\mathbf{GL}(2,\mathbf{F})}(\chi_o)=\mathrm{ind}_{\mathbf{U}}^{\mathbf{R}}(\psi.\chi_o)=\mathrm{ind}_{\mathbf{U}}^{\mathbf{R}}(\chi_{\mathbf{U}}). \tag{2.1}$$

The left-hand side of equation (1.1) can be written as

$$B[\pi(r)v,\rho(g)w]=B[\pi(r)v,\overline{\psi}(X).\rho(g)(\psi(X)w)], \tag{2.2}$$

because the character ψ_o is unitary and so is the character ψ.

Let $\rho'_{\mathbf{R}}$ be another representation of $\mathbf{R}$ which is extended from the representation ρ on $\mathbf{GL}(2,\mathbf{F})$ by the character $\overline{\psi}$ as: $\rho'_{\mathbf{R}}(r)=\overline{\psi}(X).\rho(g)$. We will have a result which is similar to (2.1) above:

$$\rho'_{\mathbf{R}}=\overline{\psi}.\,\mathrm{ind}_{\mathbf{B}_2}^{\mathbf{GL}(2,\mathbf{F})}(\chi_o)=\mathrm{ind}_{\mathbf{U}}^{\mathbf{R}}(\overline{\psi}.\chi_o).$$

Thus the right-hand side of (2.2) is

$$\begin{aligned} B[\pi(r)v,\overline{\psi}(X).\rho(g)(\psi(X).w)]&=B[\pi(r)v,\rho'_{\mathbf{R}}(r)(\psi(X).w)]\\ &=\psi(X).B[\pi(r)v,\rho'_{\mathbf{R}}(r)w]. \end{aligned} \tag{2.3}$$

From (1.1), (2.2) and (2.3), we have

$$B[\pi(r)v,\rho'_{\mathbf{R}}(r)w]=B(v,w). \tag{2.4}$$

Let $\mathcal{B}$ be the space of all bilinear forms B. Then

$\mathcal{B}\simeq \mathrm{Hom}_{\mathbf{R}}(\pi\otimes \mathrm{ind}_{\mathbf{U}}^{\mathbf{R}}(\overline{\psi}.\chi_o),\mathbb{C})$

$\simeq \mathrm{Hom}_{\mathbf{U}}(\pi_{\mathbf{U}}\otimes(\overline{\psi}.\chi_o),\mathbb{C})$ (by the Frobenius reciprocity theorem)

$\simeq \mathrm{Hom}_{\mathbf{U}}(\pi_{\mathbf{U}},\widehat{\overline{\psi}.\chi_o})$ (where $\pi_{\mathbf{U}}$ is the restriction of π on the subgroup $\mathbf{U}$ and the symbol "$\widehat{\ }$" means the contragredient)

$\simeq \mathrm{Hom}_{\mathbf{U}}(\pi_{\mathbf{U}},\psi.\chi'_o)$ (where $\chi'_o=\delta_{\mathbf{B}_2}^{s'}$ and $s'=1-s$)

$\simeq \mathrm{Hom}_{\mathbf{U}}(\pi_{\mathbf{U}},\psi.\chi_o)$.

(By exchanging $s \leftrightarrow s'$, we can replace χ_o' with χ_o.)

Let $\mathbf{U}_1 = \left\{ u_1 \in \mathbf{GSp}(6,\mathbf{F}) |\ u_1 = \begin{pmatrix} I & X' & Y \\ & I & X \\ & & I \end{pmatrix} \begin{pmatrix} b_1 & & \\ & b_1 & \\ & & b_1 \end{pmatrix},\ b_1 \in \mathbf{B}_2{}' \right\},$

where $\mathbf{B}_2{}'$ is the subgroup of $\mathbf{B}_2$, consisting of matrices of the form $b_1 = \begin{pmatrix} * & * \\ & 1 \end{pmatrix}$.

Let $\chi = \mathrm{Res}^{\mathbf{U}}_{\mathbf{U}_1} \chi_o$, the restriction on $\mathbf{U}_1$ of character χ_o of $\mathbf{U}$. (We will use the notation Res in this usual sense for the rest of this paper.)

We have $\mathbf{U} = \mathbf{U}_1.\mathbf{Z}$, where $\mathbf{Z}$ is the center of $\mathbf{U}$. The subgroup $\mathbf{Z}$ consists of diagonal matrices of the form $\mathrm{diag}(z,z,z,z,z,z)$ in $\mathbf{GSp}(6,\mathbf{F})$ (i.e. $z \in \mathbf{F}^\times$). Then

$$\begin{aligned} \mathcal{B} &\simeq \mathrm{Hom}_{\mathbf{U}_1}(\pi_{\mathbf{U}_1}, (\psi.\chi)) \quad \text{(because the two central characters match)} \\ &\simeq \mathrm{Hom}_{\mathbf{P}_6}(\pi_{\mathbf{P}_6}, \mathrm{ind}^{\mathbf{P}_6}_{\mathbf{U}_1}(\psi.\chi)) \quad \text{(by the Frobenius reciprocity theorem)} \end{aligned}$$

where $\pi_{\mathbf{P}_6}$ is the restriction of π on $\mathbf{P}_6$ which is a subgroup consisting of matrices of the form $\begin{pmatrix} * & * & * \\ & \mathbf{GSp}(4,\mathbf{F}) & * \\ & & 1 \end{pmatrix}$ in $\mathbf{P}^{1,2}$.

The group $\mathbf{P}^{1,2}$ is the standard maximal parabolic subgroup of the symplectic group $\mathbf{GSp}(6,\mathbf{F})$ whose Levi factor is isomorphic to $\mathbf{GL}(1,\mathbf{F}) \times \mathbf{GSp}(4,\mathbf{F})$. These subgroups will be described explicitly in the next section.

Thus the problem can be reduced to proving that:

(2.5) *When π is an irreducible, smooth and generic representation of* $\mathbf{GSp}(6,\mathbf{F})$, *the dimension of the space* $\mathrm{Hom}_{\mathbf{P}_6}(\pi_{\mathbf{P}_6}, \mathrm{ind}^{\mathbf{P}_6}_{\mathbf{U}_1}(\psi.\chi))$ *is at most* 1.

Now, we will replace $\pi_{\mathbf{P}_6}$ simply by π for the rest of this chapter.

3. More notations

- A. We use the notation $(\xi, \mathbf{G}, V_\xi)$ to denote a smooth (algebraic) representation ξ of a group $\mathbf{G}$ in the space V_ξ, $\xi\colon \mathbf{G} \longrightarrow \mathrm{End}(V_\xi)$. Then $\mathrm{Alg}(\mathbf{G})$ represents the category whose objects are $(\xi, \mathbf{G}, V_\xi)$ (or sometimes, just simply ξ, or (ξ, V_ξ), if no confusion arises) and whose morphisms are usual intertwining operators.

Let χ_T be a character of some abelian subgroup T of $\mathbf{G}$. Then $(\xi, \chi_T, \mathbf{G}, V_\xi)$ will denote a smooth (algebraic) representation ξ of a group $\mathbf{G}$ in the space V_ξ, $\xi\colon \mathbf{G} \longrightarrow \mathrm{End}(V_\xi)$ and $\xi(t)v = \chi_T(t)v$ for all v in V_ξ and t in T. Then the corresponding category is denoted by $\mathrm{Alg}(\mathbf{G}, \chi_T)$ whose objects are $(\xi, \chi_T, \mathbf{G}, V_\xi)$ and whose morphisms are usual intertwining operators.

When the subgroup T has only the identity of $\mathbf{G}$, we can ignore χ_T and simply write the category as $\mathrm{Alg}(\mathbf{G})$ and the object as $(\xi, \mathbf{G}, V_\xi)$.

•B. Let

$$\mathbf{GSp}(4,\mathbf{F}) = \left\{ g \in \mathbf{GL}(4,\mathbf{F}) \mid g.J_4.g^T = \mu(g).J_4 \text{ for some scalar } \mu(g) \text{ in } \mathbf{F}^\times \right\},$$

where

$$J_4 = \begin{pmatrix} & J \\ J & \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Let $\mathbf{Sp}(4,\mathbf{F})$ be its subgroup consisting of matrices of determinant 1.

Let $\mathbf{G}_o = \{\mathrm{diag}(*,*,*,1,1,1) \in \mathbf{GSp}(6,\mathbf{F})\}$, and $\mathbf{G}_2$ and $\mathbf{G}_4$ be the trivial embeddings of the groups $\mathbf{GL}(2,\mathbf{F})$ and $\mathbf{GSp}(4,\mathbf{F})$ into the group $\mathbf{G}_6 \overset{\text{def}}{=} \mathbf{GSp}(6,\mathbf{F})$:

$$\mathbf{G}_2 = \begin{pmatrix} * & & & & \\ & * & & & \\ & & \mathbf{GL}(2,\mathbf{F}) & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{S}_2 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \mathbf{SL}(2,\mathbf{F}) & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix};$$

$$\mathbf{G}_4 = \begin{pmatrix} * & & \\ & \mathbf{GSp}(4,\mathbf{F}) & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{S}_4 = \begin{pmatrix} 1 & & \\ & \mathbf{Sp}(4,\mathbf{F}) & \\ & & 1 \end{pmatrix}.$$

•C. We will define the subgroups $\mathbf{P}_n$ of the standard maximal parabolic subgroups of $\mathbf{G}_n$, $n = 0, 2, 4, 6$. Let

$$\mathbf{T}_6 = \left\{ t \mid t = \begin{pmatrix} 1 & x_1 & -x_2 & x_3 & -x_4 & x_5 \\ & 1 & & & & x_4 \\ & & 1 & & & x_3 \\ & & & 1 & & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix} \right\},$$

$$\mathbf{T}_4 = \left\{ t \mid t = \begin{pmatrix} 1 & & & & & \\ & 1 & y_1 & -y_2 & y_3 & \\ & & 1 & & y_2 & \\ & & & 1 & y_1 & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\},$$

$$\mathbf{T}_2 = \left\{ t \mid t = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & z & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\}.$$

All x_i, y_j, z are in $\mathbf{F}$. Then $\mathbf{P}_6 = \mathbf{G}_4.\mathbf{T}_6$; $\mathbf{P}_4 = \mathbf{G}_2.\mathbf{T}_4$; $\mathbf{P}_2 = \mathbf{G}_o.\mathbf{T}_2$ is the embedding of $\mathbf{B}_2{}'$ into $\mathbf{GSp}(6,\mathbf{F})$:

$$\mathbf{P}_2 = \left\{ p \mid p = \begin{pmatrix} * & & & & & \\ & * & & & & \\ & & * & z & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathbf{P}_o \equiv 1.$$

•D. Let $\mathbf{L}_n$ be the center subgroups of $\mathbf{T}_n$, where $\mathbf{L}_2 = \mathbf{T}_2$,

$$\mathbf{L}_4 = \left\{ l \mid l = \begin{pmatrix} 1 & & & & & \\ & 1 & & & y_3 & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\}$$

and

$$\mathbf{L}_6 = \left\{ l \mid l = \begin{pmatrix} 1 & & & & & x_5 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\}.$$

Let $\mathbf{K}_6$ be the quotient subgroup $\mathbf{T}_6/\mathbf{L}_6$. Then

$$\mathbf{K}_6 = \left\{ \bar{k} \mid \bar{k} = \mathbf{L}_6.k, \text{ where } k = \begin{pmatrix} 1 & x_1 & -x_2 & x_3 & -x_4 & 0 \\ & 1 & & & & x_4 \\ & & 1 & & & x_3 \\ & & & 1 & & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix} \right\}.$$

$\mathbf{K}_6$ is equivalent to a four-dimensional abelian subgroup of $\mathbf{T}_6$. We have $\mathbf{K}_6 \simeq \mathbf{F}^4$.

Similarly, let $\mathbf{K}_4$ be the quotient subgroup $\mathbf{T}_4/\mathbf{L}_4$. Then

$$\mathbf{K}_4 = \left\{ \bar{k} \mid \bar{k} = \mathbf{L}_4.k, \text{ where } k = \begin{pmatrix} 1 & & & & & \\ & 1 & y_1 & -y_2 & 0 & \\ & & 1 & & y_2 & \\ & & & 1 & y_1 & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\}.$$

$\mathbf{K}_4$ is equivalent to a two-dimensional abelian subgroup of $\mathbf{T}_4$. We have $\mathbf{K}_4 \simeq \mathbf{F}^2$.

4. The functors for the subgroup $\mathbf{P}_6$

• A. We will construct the characters for $\mathbf{L}_6$ and $\mathbf{K}_6$. Since they are isomorphic to $\mathbf{F}$ and $\mathbf{F}^4$, respectively, we can recall lemma 5.4 in chapter III of [B,Z], with reference to the notations introduced in the above section 3.D.

Let

$$t = \begin{pmatrix} 1 & x_1 & -x_2 & x_3 & -x_4 & x_5 \\ & 1 & & & & x_4 \\ & & 1 & & & x_3 \\ & & & 1 & & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix} \in \mathbf{T}_6.$$

Then we have the identity: $g.t.g^{-1} = {}^g t$, for $g \in \mathbf{G}_4$ and

$$g = \begin{pmatrix} p_{11} & & & & & \\ & p_{22} & p_{23} & p_{24} & p_{25} & \\ & p_{32} & p_{33} & p_{34} & p_{35} & \\ & p_{42} & p_{43} & p_{44} & p_{45} & \\ & p_{52} & p_{53} & p_{54} & p_{55} & \\ & & & & & 1 \end{pmatrix}$$

and

$${}^g t = \begin{pmatrix} 1 & w_1 & -w_2 & w_3 & -w_4 & w_5 \\ & 1 & & & & w_4 \\ & & 1 & & & w_3 \\ & & & 1 & & w_2 \\ & & & & 1 & w_1 \\ & & & & & 1 \end{pmatrix},$$

where $w_5 = p_{11}.x_5$ and

(4.1)

$$w_4 = p_{22}x_4 + p_{23}x_3 + p_{24}x_2 + p_{25}x_1; \quad w_3 = p_{32}x_4 + p_{33}x_3 + p_{34}x_2 + p_{35}x_1;$$

$$w_2 = p_{42}x_4 + p_{43}x_3 + p_{44}x_2 + p_{45}x_1; \quad w_1 = p_{52}x_4 + p_{53}x_3 + p_{54}x_2 + p_{55}x_1.$$

LEMMA 4.1: *Define $\Theta_{\mathbf{L},1}(l) = \psi_o(x_5)$, and $\Theta_{\mathbf{L},o}(l) = 1$, for all l in $\mathbf{L}_6$. Then any non-trivial character $\Theta_{\mathbf{L}}$ of $\mathbf{L}_6$ is conjugate to $\Theta_{\mathbf{L},1}$ under the action of $\mathbf{P}_6$.*

Proof: Any non-trivial character $\Theta_{\mathbf{L}}$ of $\mathbf{L}_6$ is of the form $\Theta_{\mathbf{L}}(l) = \psi_o(a.x_5)$ for some $a \in \mathbf{F}^\times$. Let $g \in \mathbf{P}_6$ as described above with $p_{11} = a$. Then (4.1) gives us

$$g.l.g^{-1} = {}^g l = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a.x_5 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Thus $\Theta_{\mathbf{L},1}({}^g l) = \psi_o(a.x_5) = \Theta_{\mathbf{L}}(l)$. ∎

Therefore, we will be interested only in the two characters $\Theta_{\mathbf{L},o}$ and $\Theta_{\mathbf{L},1}$ in arguments in parts B and C below. We have a similar lemma for $\mathbf{K}_6$.

LEMMA 4.2: *Define $\Theta_{\mathbf{K},1}(\bar{k}) = \Theta_{\mathbf{K},1}(k) = \psi_o(x_1)$, and $\Theta_{\mathbf{K},o}(\bar{k}) = \Theta_{\mathbf{K},o}(k) = 1$, for all $\bar{k}$ in $\mathbf{K}_6$. Then any non-trivial character of $\mathbf{K}_6$ is conjugate to $\Theta_{\mathbf{K},1}$ under the action of $\mathbf{P}_6$.*

Proof: Any non-trivial character $\Theta_{\mathbf{K}}$ of $\mathbf{K}_6$ is of the form

$$\Theta_{\mathbf{K}}(\bar{k}) = \psi_o(a_1.x_1 + a_2.x_2 + a_3.x_3 + a_4.x_4) \quad \text{for } a_i \in \mathbf{F}.$$

Let $g \in \mathbf{P}_6$ as described above with $p_{52} = a_4$, $p_{53} = a_3$, $p_{54} = a_2$ and $p_{55} = a_1$. Then (4.1) gives us

$$g.\bar{k}.g^{-1} = \bar{k}_1 \quad \text{where } k_1 = \begin{pmatrix} 1 & w_1 & -w_2 & w_3 & -w_4 & 0 \\ & 1 & & & & w_4 \\ & & 1 & & & w_3 \\ & & & 1 & & w_2 \\ & & & & 1 & w_1 \\ & & & & & 1 \end{pmatrix},$$

where $w_1 = p_{52}x_4 + p_{53}x_3 + p_{54}x_2 + p_{55}x_1$. Therefore,

$$\Theta_{\mathbf{K},1}(\bar{k}_1) = \psi_o(w_1) = \psi_o(p_{52}x_4 + p_{53}x_3 + p_{54}x_2 + p_{55}x_1) = \Theta_{\mathbf{K}}(\bar{k}). \quad ∎$$

Thus, in part D below, we will consider only two characters $\Theta_{\mathbf{K},o}$ and $\Theta_{\mathbf{K},1}$.

•B. The normalizers of the two characters $\Theta_{\mathbf{L},i}$, for $i = 0, 1$, are defined as

$$\text{Norm}_{\mathbf{P}_6}(\mathbf{L}_6, \Theta_{\mathbf{L},i}) \stackrel{\text{def}}{=} \left\{ p \in \mathbf{P}_6 \mid p.l.p^{-1} \in \mathbf{L}_6, \text{ and } \Theta_{\mathbf{L},i}(p.l.p^{-1}) = \Theta_{\mathbf{L},i}(l), \text{ for all } l \in \mathbf{L}_6 \right\}.$$

LEMMA 4.3:

$$(\mathbf{G}_4.\mathbf{T}_6) = \text{Norm}_{\mathbf{P}_6}(\mathbf{L}_6, \Theta_{\mathbf{L},o}) \quad \textit{and} \quad (\mathbf{S}_4.\mathbf{T}_6) = \text{Norm}_{\mathbf{P}_6}(\mathbf{L}_6, \Theta_{\mathbf{L},1}).$$

Proof: $\mathbf{L}_6$ is an abelian subgroup of $\mathbf{T}_6$. To compute the action by conjugation of $(\mathbf{G}_4.\mathbf{T}_6) = \mathbf{P}_6$ on $\mathbf{L}_6$, it suffices to consider only that action of $\mathbf{G}_4$ on $\mathbf{T}_6$. From the above definition and results in (4.1),

– For $\Theta_{\mathbf{L},o} \equiv 1$, trivially, its normalizer is the subgroup $(\mathbf{G}_4.\mathbf{T}_6) = \mathbf{P}_6$.

– For $\Theta_{\mathbf{L},1}(l) = \psi_o(x_5)$, we need: $\psi_o(w_5) = \psi_o(p_{11}x_5)$ for all x_5. Therefore, $p_{11} = 1$ is required. That is, the normalizer is the subgroup $(\mathbf{S}_4.\mathbf{T}_6)$. ∎

•C. The functors corresponding to the two characters $\Theta_{\mathbf{L},i}$, for $i = 0, 1$.

We can define the Jacquet functors $\mathcal{J}_-^{\mathbf{L},i}$ as follows:

$$-\mathcal{J}_-^{\mathbf{L},o}\colon \mathrm{Alg}(\mathbf{P}_6) \longrightarrow \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o}) \quad \text{and}$$
$$-\mathcal{J}_-^{\mathbf{L},1}\colon \mathrm{Alg}(\mathbf{P}_6) \longrightarrow \mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1}).$$

Let $(\pi, \mathbf{P}_6, V_\pi)$ be an object in $\mathrm{Alg}(\mathbf{P}_6)$. Then, correspondingly,

- $(\mathcal{J}_-^{\mathbf{L},o}(\pi), \Theta_{\mathbf{L},o}, (\mathbf{G}_4.\mathbf{K}_6), V_{\mathbf{L}_6,\Theta_o})$ is an object in $\mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$ and
- $(\mathcal{J}_-^{\mathbf{L},1}(\pi), \Theta_{\mathbf{L},1}, (\mathbf{S}_4.\mathbf{T}_6), V_{\mathbf{L}_6,\Theta_1})$ is an object in $\mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1})$, where

$$V_{\mathbf{L}_6,\Theta_i} = V_\pi / V(\mathbf{L}_6, \Theta_{\mathbf{L},i})$$

and

$$V(\mathbf{L}_6, \Theta_{\mathbf{L},i}) = \Big\langle \pi(l)v - \Theta_{\mathbf{L},i}(l)v, \text{ for all } v \in V_\pi, l \in \mathbf{L}_6 \Big\rangle.$$

We can also define the functors $\mathcal{J}_+^{\mathbf{L},i}$:

$$-\mathcal{J}_+^{\mathbf{L},o}\colon \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o}) \longrightarrow \mathrm{Alg}(\mathbf{P}_6).$$

Let $(\theta_o, \Theta_{\mathbf{L},o}, (\mathbf{G}_4.\mathbf{T}_6), V'_o)$ be an object in $\mathrm{Alg}(\mathbf{G}_4.T_6, \Theta_{\mathbf{L},o})$. Then $(\mathcal{J}_+^{\mathbf{L},o}(\theta_o), \mathbf{P}_6, V')$ is an object in $\mathrm{Alg}(\mathbf{P}_6)$, where we also have $\mathcal{J}_+^{\mathbf{L},o}(\theta_o)((g.t).l)v' = \Theta_{\mathbf{L},o}(l).\theta_o(g.t)v' = \theta_o(g.t)v'$, for all $(g.t) \in (\mathbf{G}_4.\mathbf{T}_6)$ and $l \in \mathbf{L}_6$, and $v' \in V'_o$. That is, $\mathcal{J}_+^{\mathbf{L},o}$ is just an embedding $\mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o}) \hookrightarrow \mathrm{Alg}(\mathbf{P}_6)$.

$$-\mathcal{J}_+^{\mathbf{L},1}\colon \mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1}) \longrightarrow \mathrm{Alg}(\mathbf{P}_6).$$

Let $(\theta_1, \Theta_{\mathbf{L},1}, (\mathbf{S}_4.\mathbf{T}_6), V'_1)$ be an object in $\mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1})$. Then $(\mathcal{J}_+^{\mathbf{L},1}(\theta_1), \mathbf{P}_6, V')$ is an object in $\mathrm{Alg}(\mathbf{P}_6)$, where $\mathcal{J}_+^{\mathbf{L},1}(\theta_1) = \mathrm{ind}_{(\mathbf{S}_4.\mathbf{T}_6)}^{\mathbf{P}_6}(\theta_1)$ is an unnormalized compact induction; and $(\theta_1, \Theta_{\mathbf{L},1}, (\mathbf{S}_4.\mathbf{T}_6), V'_1)$ satisfies: $\theta_1((s.t).l)v' = \Theta_{\mathbf{L},1}(l).\theta_1(s.t)v'$, for all $v' \in V'_1$, $(s.t) \in (\mathbf{S}_4.\mathbf{T}_6)$ and $l \in \mathbf{L}_6$.

•D. We have the same results for the normalizers of two characters $\Theta_{\mathbf{K},i}$.

$$\mathrm{Norm}_{(\mathbf{G}_4.\mathbf{T}_6)}(\mathbf{K}_6, \Theta_{\mathbf{K},i}) \stackrel{\text{def}}{=}$$
$$\Big\{ p \in (\mathbf{G}_4.\mathbf{T}_6) \mid p.\bar{k}.p^{-1} \in \mathbf{K}_6, \text{ and } \Theta_{\mathbf{K},i}(p.\bar{k}.p^{-1}) = \Theta_{\mathbf{K},i}(\bar{k}), \text{ for all } \bar{k} \in \mathbf{K}_6 \Big\}$$
$$= (\mathbf{M}_i.\mathbf{T}_6), \text{ where}$$
$$\mathbf{M}_i = \Big\{ g \in \mathbf{G}_4 \mid g.\bar{k}.g^{-1} \in \mathbf{K}_6, \text{ and } \Theta_{\mathbf{K},i}(g.\bar{k}.g^{-1}) = \Theta_{\mathbf{K},i}(\bar{k}), \text{ for all } \bar{k} \in \mathbf{K}_6 \Big\}.$$

LEMMA 4.4:

$$(\mathbf{G}_4.\mathbf{T}_6) = \mathrm{Norm}_{(\mathbf{G}_4.\mathbf{T}_6)}(\mathbf{K}_6, \Theta_{\mathbf{K},o}) \quad \textit{and} \quad (\mathbf{P}_4.\mathbf{T}_6) = \mathrm{Norm}_{(\mathbf{G}_4.\mathbf{T}_6)}(\mathbf{K}_6, \Theta_{\mathbf{K},1}).$$

We now define the functors corresponding to the two characters $\Theta_{\mathbf{K},i}$, for $i = 0, 1$, acting on $\mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$.

We ignore the functors acting on $\mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1})$ for certain reasons which will be seen later. (Also refer to the chart in Section 0. Introduction.)

We can define the Jacquet functors $\mathcal{J}_-^{\mathbf{K},i}$ as follows:

$$-\mathcal{J}_-^{\mathbf{K},o}\colon \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o}) \longrightarrow \mathrm{Alg}(\mathbf{G}_4) \quad \text{and}$$
$$-\mathcal{J}_-^{\mathbf{K},1}\colon \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o}) \longrightarrow \mathrm{Alg}(\mathbf{P}_4).$$

Let $(\theta, \Theta_{\mathbf{L},o}, (\mathbf{G}_4.\mathbf{T}_6), V_\theta)$ be an object in $\mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$. Then, correspondingly,

- $(\mathcal{J}_-^{\mathbf{K},o}(\theta), \mathbf{G}_4, V_{\mathbf{K}_6,\Theta_o})$ is an object in $\mathrm{Alg}(\mathbf{G}_4)$ and
- $(\mathcal{J}_-^{\mathbf{K},1}(\theta), \mathbf{P}_4, V_{\mathbf{K}_6,\Theta_1})$ is an object in $\mathrm{Alg}(\mathbf{P}_4)$, where $V_{\mathbf{K}_6,\Theta_i} = V_\theta/V(\mathbf{K}_6, \Theta_{\mathbf{K},i})$ and

$$V(\mathbf{K}_6, \Theta_{\mathbf{K},i}) = \Big\langle \theta(\bar{k})v - \Theta_{\mathbf{K},i}(\bar{k})v, \text{ for all } v \in V_\theta \text{ and } \bar{k} \in \mathbf{K}_6 \Big\rangle.$$

Let us denote $\mathrm{Alg}(\mathbf{G}_4.\mathbf{K}_6)$ to be a sub-category of $\mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$, consisting of representations which act trivially (by $\Theta_{\mathbf{L},o}$) on $\mathbf{L}_6$. That is, we identify $(\theta, \Theta_{\mathbf{L},o}, (\mathbf{G}_4.\mathbf{T}_6), V_\theta)$ with $(\theta, (\mathbf{G}_4.\mathbf{K}_6), V_\theta)$.

We can also define the functors $\mathcal{J}_+^{\mathbf{K},i}$:

$$-\mathcal{J}_+^{\mathbf{K},o}\colon \mathrm{Alg}(\mathbf{G}_4) \longrightarrow \mathrm{Alg}(\mathbf{G}_4.\mathbf{K}_6).$$

Let $(\sigma_o, \mathbf{G}_4, V_o')$ be an object in $\mathrm{Alg}(\mathbf{G}_4)$. Then $(\mathcal{J}_+^{\mathbf{K},o}(\sigma_o), (\mathbf{G}_4.\mathbf{K}_6), V')$ is an object in $\mathrm{Alg}(\mathbf{G}_4.\mathbf{K}_6)$.

$$-\mathcal{J}_+^{\mathbf{K},1}\colon \mathrm{Alg}(\mathbf{P}_4) \longrightarrow \mathrm{Alg}(\mathbf{G}_4.\mathbf{K}_6).$$

Let $(\sigma_1, \mathbf{P}_4, V_1')$ be an object in $\mathrm{Alg}(\mathbf{P}_4)$. Then $(\mathcal{J}_+^{\mathbf{K},1}(\sigma_1), (\mathbf{G}_4.\mathbf{K}_6), V')$ is an object in $\mathrm{Alg}(\mathbf{G}_4.\mathbf{K}_6)$, where $\mathcal{J}_+^{\mathbf{K},i}(\sigma_i) = \mathrm{ind}_{(\mathbf{M}_i.\mathbf{K}_6)}^{(\mathbf{G}_4.\mathbf{K}_6)}(\sigma_i')$ is an unnormalized compact induction and $(\sigma_i', (\mathbf{M}_i.\mathbf{K}_6), V_i') \in \mathrm{Alg}(\mathbf{M}_i.\mathbf{K}_6)$ is defined by $\sigma_i'(m.\bar{k})v' = \Theta_{\mathbf{K},i}(\bar{k}).\sigma_i(m)v'$, for all $v' \in V_i'$, $m \in \mathbf{M}_i$, $\bar{k} \in \mathbf{K}_6$, where we denote $\mathbf{M}_o = \mathbf{G}_4$ and $\mathbf{M}_1 = \mathbf{P}_4$. (Obviously, the induction inside the functor $\mathcal{J}_+^{\mathbf{K},o}$ is identity.)

5. The proposition

Following [B,Z], we state some properties of the functors $\mathcal{J}_-^{\mathbf{L},i}$, $\mathcal{J}_+^{\mathbf{L},i}$, $i = 0, 1$.

PROPOSITION I: *For any representation* $\pi \in \mathrm{Alg}(\mathbf{P}_6)$, $\theta_1 \in \mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1})$, *and* $\theta_o \in \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$ *and* $i = 0, 1$, *we have:*

(1) *All the functors* $\mathcal{J}_-^{\mathbf{L},i}, \mathcal{J}_+^{\mathbf{L},i}$ *are exact.*

(2) $\mathcal{J}_-^{\mathbf{L},i}\mathcal{J}_+^{\mathbf{L},i}(\theta_i) \simeq \theta_i$ *and* $\mathcal{J}_-^{\mathbf{L},j}\mathcal{J}_+^{\mathbf{L},i}(\theta_i) \simeq 0$ *if* $i \neq j$.

(3) $\mathcal{J}_+^{\mathbf{L},1}$ *is left-adjoint to* $\mathcal{J}_-^{\mathbf{L},1}$; *that is, there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},1}(\theta_1), \pi) \simeq \mathrm{Hom}_{(\mathbf{S}_4.\mathbf{T}_6)}(\theta_1, \mathcal{J}_-^{\mathbf{L},1}(\pi)) \tag{5.1}$$

which depends functorially on π *and* θ_1.

$\mathcal{J}_-^{\mathbf{L},o}$ *is right-adjoint to* $\mathcal{J}_+^{\mathbf{L},o}$; *that is, there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathcal{J}_+^{\mathbf{L},o}(\theta_o)) \simeq \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\mathcal{J}_-^{\mathbf{L},o}(\pi), \theta_o) \tag{5.2}$$

which depends functorially on π *and* θ_o.

(4) *Let us consider the homomorphisms:*

$$\ell_o\colon \pi \longrightarrow \mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi);\quad \ell_o'\colon \mathcal{J}_-^{\mathbf{L},o}\mathcal{J}_+^{\mathbf{L},o}(\theta_o) \longrightarrow \theta_o,$$
$$\ell_1\colon \mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi) \longrightarrow \pi;\quad \ell_1'\colon \theta_1 \longrightarrow \mathcal{J}_-^{\mathbf{L},1}\mathcal{J}_+^{\mathbf{L},1}(\theta_1).$$

Then ℓ_o' *and* ℓ_1' *are isomorphisms, and* ℓ_o *and* ℓ_1 *form a short exact sequence:*

$$0 \longrightarrow \mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi) \xrightarrow{\ell_1} \pi \xrightarrow{\ell_o} \mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi) \longrightarrow 0. \tag{5.3}$$

(5) $\mathcal{J}_-^{\mathbf{L},i}$ *and* $\mathcal{J}_+^{\mathbf{L},i}$ *establish a bijection between* θ_i *and* $\mathcal{J}_+^{\mathbf{L},i}(\theta_i)$. *In particular,* θ_i *and* $\mathcal{J}_+^{\mathbf{L},i}(\theta_i)$ *are irreducible simultaneously.*

The proof of Proposition I will be given in Section 9.

We have the same proposition for the functors $\mathcal{J}_-^{\mathbf{K},i}$ and $\mathcal{J}_+^{\mathbf{K},i}$.

6. Setting up the double coset calculation

We reduced our work to investigating only the space $\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))$ (by 2.5.) From Proposition I, part 1 proves that all the functors $\mathcal{J}_-^{\mathbf{L},i}, \mathcal{J}_+^{\mathbf{L},i}$, where $i = 0, 1$, are exact. Therefore, it together with the short exact sequence in part 4 will give us this short exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \longrightarrow \mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))$$
$$\longrightarrow \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)).$$

Eventually, we will prove that the space $\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))$ is trivial and $\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))$ is at most one-dimensional.

For $i = 0, 1$, let $\theta_i = \mathcal{J}_-^{\mathbf{L},i}(\pi)$. Then we will consider only the following spaces:

$$\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},i}\mathcal{J}_-^{\mathbf{L},i}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \simeq \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},i}(\theta_i), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)).$$

In fact, we will investigate the space $\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},i}(\theta_i), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))$ in the following lemma for any representations θ_i, where $\theta_1 \in \mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1})$ and $\theta_o \in \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$ and $\mathcal{J}_+^{\mathbf{L},i}(\theta_i)$ was defined in Section 4.C.

We need the following lemma, which will play a basic role in double coset calculation in the next sections.

LEMMA 6.1: *Let X be a locally compact, totally disconnected group, and H_1, H_2 be its closed subgroups. Let $(\sigma_1, H_1, V_{\sigma_1})$ be a representation of H_1 and χ_2 be a character of H_2. Let the group $H = H_1 \times H_2$ act on the group X by: $x \mapsto h_2 x h_1^{-1}$. Assume this action is constructible.*† *For $x \in X$, let $y_x = H_2 x H_1$ be a double coset representative in $Y = H_2\backslash X/H_1$. Let $\chi_2^x(h) = \chi_2(xhx^{-1})$, for all $h \in {}^xH = H_1 \cap x^{-1}H_2x$. If the space $\mathrm{Hom}_{({}^xH)}(\sigma_1, \chi_2^x) = 0$ for all orbits y_x but only one orbit $y_{x_o} = H_2 x_o H_1$, then*

$$\mathrm{Hom}_X(\sigma_1^X, \chi_2^X) \hookrightarrow \mathrm{Hom}_{({}^{x_o}H)}(\sigma_1, \chi_2^{x_o}), \tag{6.1}$$

where the inductions are compact and not normalized.

† *Remark:* In fact, by the Appendix in [B,Z], p. 62, one can prove this assumption holds in applications in this chapter.

Proof: We recall that $Y = H_2\backslash X/H_1$. Then Y is generally not Hausdorff. We define the projection by double coset, $P\colon X \longrightarrow Y$, by $x \mapsto y_x = H_2 x H_1$. It is a continuous mapping.

Let us recall the action of group $H = (H_1 \times H_2)$ on group X by $x \mapsto h_2 x h_1^{-1}$. Then the orbit of $x \in X$ is just the double coset $H_2 x H_1$. Its stablizer is calculated simply, $\mathrm{stab}(x) = H_1 \cap x^{-1}H_2x$, and now denoted as ${}^xH$ which is embedded into the group $H = (H_1 \times H_2)$ by $h \mapsto (h, xhx^{-1})$ for any $h \in {}^xH$. Then $(H_1 \times H_2)/{}^xH \simeq H_2 x H_1$.

We now consider two representations on ${}^xH$ which are the restrictions of the representation σ_1 and of the conjugated character χ_2^x, which is defined by

$$\chi_2^x(h) = \chi_2(xhx^{-1}), \quad \text{for all } h \in {}^xH.$$

Let us define a representation (σ, H, V_σ) on the group $H = (H_1 \times H_2)$. Let $V_\sigma = \mathrm{Hom}_{\mathbb{C}}(V_{\sigma_1}, V_{\chi_2})$. For $T \in V_\sigma$, $v \in V_{\sigma_1}$, let

$$\sigma(h_1, h_2)T(v) = \chi_2(h_2)^{-1}\, T(\sigma_1(h_1)v).$$

On the subgroup ${}^xH$,

$$\sigma(h_1, xh_1x^{-1})T(v) = \chi_2(xh_1x^{-1})^{-1}.T(\sigma_1(h_1)v) = \chi_2^x(h_1)^{-1}.T(\sigma_1(h_1)v),$$

for all $h_1 \in {}^xH$. Thus

$$\sigma(h_1, xh_1x^{-1})T(v) = T(v) \quad \text{is equivalent to} \quad T(\sigma_1(h_1)v) = \chi_2^x(h_1)T(v).$$

Thus we have a natural isomorphism:

$$\mathrm{Hom}_{({}^xH)}(1_{V_{\sigma_1}}, \sigma) \simeq \mathrm{Hom}_{({}^xH)}(\sigma_1, \chi_2^x). \tag{6.2}$$

- We now consider another representation of H, $(\sigma^\star, H, C_c^\infty(X, V_{\sigma_1}))$, defined by

  $$\sigma^\star(h)f(x) = \sigma^\star(h_1, h_2)f(x) = \chi_2(h_2^{-1}).\sigma_1(h_1)f(x),$$

  for any $x \in X$, $h = (h_1, h_2) \in H$ and for any $f \in C_c^\infty(X, V_{\sigma_1})$, f is a compactly supported smooth function: $X \longrightarrow V_{\sigma_1}$. Then the representation $\sigma^\star$ inherits the same property (6.2); that is, there exists a natural isomorphism

  $$\mathrm{Hom}_{({}^xH)}(1_{V_{\sigma_1}}, \sigma^\star) \simeq \mathrm{Hom}_{({}^xH)}(\sigma_1, \chi_2^x). \tag{6.3}$$

  We also define the action of representation $\sigma^\star$ on $\mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X, V_{\sigma_1}), V_{\chi_2}) \simeq \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X, V_{\sigma_1}), \mathbb{C})$ as $\sigma^\star\colon H \longrightarrow \mathrm{End}(\mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X, V_{\sigma_1}), V_{\chi_2}))$.

For any $\Delta \in \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X, V_{\sigma_1}), V_{\chi_2})$, $f \in C_c^\infty(X, V_{\sigma_1})$ and $h \in H$,

$$(\sigma^\star(h)\Delta)(f) = \Delta(\sigma^\star(h)(f)). \tag{6.4}$$

The right translations ρ and λ act on the induced spaces σ_1^X and χ_2^X, respectively, in the usual way. Let us define the translation action of the group $H = (H_1 \times H_2)$ on the space σ_1^X by

$$\kappa(h)f = \kappa(h_1, h_2)f = \lambda(h_2)\rho(h_1)f, \quad \text{for all } f \in \sigma_1^X. \tag{6.5}$$

Let κ act on the space $\mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X, V_{\sigma_1}), V_{\chi_2})$ by

$$(\kappa(h)\Delta)(f) = \Delta(\kappa(h)f), \quad \text{for any } \Delta, f \text{ and } h \in H. \tag{6.6}$$

Let $\mathcal{D}(X)$ be the subspace of all distributions Δ in space $\mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X, V_{\sigma_1}), V_{\chi_2})$ which satisfy

$$\kappa(h)\Delta = \sigma^\star(h)\Delta, \quad \text{for all } \Delta \in \mathcal{D}(X), \quad h = (h_1, h_2) \in H. \tag{6.7}$$

- Let us define the sheaf $\mathcal{F}$ on the group X. For any open set $U \in X$, we have $\mathcal{F}(U) = C^\infty(U, V_{\sigma_1})$. Let $\mathcal{F}_c$ be the corresponding sheaf comprising a module of compactly supported sections: for any open set $U \in X$, we have $\mathcal{F}_c(U) = C_c^\infty(U, V_{\sigma_1})$.

Now let $Z = P^{-1}(y_x) = H_2 x H_1$. Then Z is a (closed) double coset in X, which is stable under the action of $H = (H_1 \times H_2)$. Let $\mathcal{F}_Z$ be the restriction of the sheaf $\mathcal{F}$ on Z, which is defined as: $\mathcal{F}_Z(U, V_{\sigma_1}) = \mathcal{F}(U, V_{\sigma_1})$ for any open subset U in Z. Then $\mathcal{F}_Z$ is also a sheaf and the fiber $\mathcal{F}_{Z,z} \simeq \mathcal{F}_z$, for any $z \in Z$. The etale space of $\mathcal{F}_Z$ is just the restriction of the etale space of $\mathcal{F}$ to Z.

Now, we also define $(\mathcal{F}_Z)_c$ as the restriction of $\mathcal{F}_c$ on the subgroup Z.

Let $\mathcal{F}_c(\sigma^\star)$ be a submodule of $\mathcal{F}_c$ generated by elements of the form $\kappa(h)f - \sigma^\star(h)f$, for $f \in \mathcal{F}_c$ and $h \in H = (H_1 \times H_2)$. Then $M = \mathcal{F}_c/\mathcal{F}_c(\sigma^\star)$ is a $C_c^\infty(Y, V_{\sigma_1})$-module.

Similarly, we define $(\mathcal{F}_Z)_c(\sigma^\star)$ to be a submodule of $(\mathcal{F}_Z)_c$ generated by elements of the form $\kappa(h)f - \sigma^\star(h)f$, for $f \in (\mathcal{F}_Z)_c$.

The proposition I-9 in [E,H] shows that it is only necessary to specify a sheaf on a base of the topology of the space. Let

$$\mathcal{B}_o = \{P(H_2 K H_1), \text{ where } K \subset X \text{ is both compact and open}\}$$

be a base of the topology of space Y. Then $\mathcal{G}$, the corresponding sheaf on Y, associated with M and base $\mathcal{B}_o$, is defined as follows:

For any subset $U \in \mathcal{B}_o$, let $\mathcal{G}(U) = \mathbf{1}_U.M$. For $U, V \in \mathcal{B}_o$ and $V \subseteq U$, the restriction map $\rho_{U,V}$: $\mathcal{G}(U) \to \mathcal{G}(V)$ is defined by $\rho_{U,V}(m) = \mathbf{1}_V.m$, for any $m \in M$. Then $\mathcal{G}$ is proven to be a presheaf which satisfies the sheaf axiom. Hence it is a sheaf.

Now the condition in theorem 2.36 of Bernstein and Zelevinsky, in [B,Z], that Y is a Hausdorff space, can be waived and its constructibility property is sufficient instead, for our purpose (by the same arguments used to prove theorem 6.9, loc. cit.). Then the theorem shows that:

(6.8) *For any $y_x \in Y$, the stalk $\mathcal{G}_{y_x}$ is isomorphic to $(\mathcal{F}_Z)_c/(\mathcal{F}_Z)_c(\sigma^\star)$, where we recall $Z = P^{-1}(y_x) = H_2 x H_1$.*

- For any $\Delta \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_x}, V_{\chi_2}) \simeq \mathrm{Hom}_{\mathbb{C}}((\mathcal{F}_Z)_c/(\mathcal{F}_Z)_c(\sigma^\star), V_{\chi_2})$, it is a distribution on the subgroup Z, which satisfies $\Delta(\kappa(h)f - \sigma^\star(h)f) = 0$, for any $f \in (\mathcal{F}_Z)_c$ and $h \in H$. Therefore, by (6.4) and (6.6),

$$(\kappa(h)\Delta)(f) - (\sigma^\star(h)\Delta)(f) = 0 \quad \text{or} \quad \kappa(h)\Delta = \sigma^\star(h)\Delta.$$

Now we need the following lemma.

Lemma 6.2: *Let (σ, H, V_{χ_2}) be any representation of the group H and H_o be its closed subgroup. Let Δ be a distribution of H/H_o, satisfying*

$$\kappa(h)\Delta = \sigma(h)\Delta, \quad \textit{for all } h \textit{ in } H. \tag{6.9}$$

If $\Delta \neq 0$, then $\Delta \in \mathrm{Hom}_{({}^xH)}(1_{V_{\sigma_1}}, \sigma)$.

Proof: Δ is a distribution on H/H_o, hence it is invariant under action of H_o. By (6.9), for any $f \in C_c^\infty(H, V_{\sigma_1})$ and $h \in H_o$, we can write

$$\sigma(h)\Delta(f) = \kappa(h)\Delta(f) = \kappa(1)\Delta(f) = \Delta(f).$$

By the condition $\Delta \neq 0$, we can choose some f, such that $\Delta(f) \neq 0$. That gives us $\Delta \in \mathrm{Hom}_{({}^xH)}(1_{V_{\sigma_1}}, \sigma)$. ∎

Let $H = (H_1 \times H_2)$, $H_o = {}^xH$ and σ is $\sigma^\star$ in the above lemma. (Thus $y_x \simeq H/H_o$.) Then for any $\Delta \neq 0$ in the space $\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_x}, V_{\chi_2})$, we proved that Δ is also in the space $\mathrm{Hom}_{({}^xH)}(1_{V_{\sigma_1}}, \sigma^\star)$. Recalling the result (6.3), we have

$$\mathrm{Hom}_{({}^xH)}(1_{V_{\sigma_1}}, \sigma^\star) \simeq \mathrm{Hom}_{({}^xH)}(\sigma_1, \chi_2^x).$$

Thus we have the following embedding:

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_x}, V_{\chi_2}) \hookrightarrow \mathrm{Hom}_{({}^xH)}(\sigma_1, \chi_2^x). \tag{6.10}$$

- On the other side, for any $\Delta \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_c/\mathcal{F}_c(\sigma^\star), V_{\chi_2})$, it is a linear functional on the space of compactly supported global sections of $\mathcal{G}$, or, in other words, a distribution on X, which satisfies $\Delta(\kappa(h)f - \sigma^\star(h)f) = 0$, for any $f \in \mathcal{F}_c$ and $h \in H$. That is, $\kappa(h)\Delta = \sigma^\star(h)\Delta$. Thus

$$\mathcal{D}(X) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}).$$

Now we try to embed $\mathrm{Hom}_X(\sigma_1^X, \chi_2^X)$ into $\mathcal{D}(X)$. From Frobenius' reciprocity theorem,

$$\mathrm{Hom}_X(\sigma_1^X, \chi_2^X) \simeq \mathrm{Hom}_{H_2}(\sigma_1^X, \chi_2).$$

Given any homomorphism $\phi \in \mathrm{Hom}_{H_2}(\sigma_1^X, \chi_2)$, we will construct a distribution Δ on X as follows:

Let Π be a projection, Π: $C_c^\infty(X, V_{\sigma_1}) \longrightarrow \sigma_1^X$. For all $f \in C_c^\infty(X, V_{\sigma_1})$, we can define $(\Pi f)(x) = \int_{H_1} \sigma_1(h)^{-1} f(hx)\, dh$. Then, for any $h_1 \in H_1$,

$$(\Pi f)(h_1 x) = \int\limits_{H_1} \sigma_1(h)^{-1} f(hh_1 x)\, dh = \int\limits_{H_1} \sigma_1(h_1)\sigma_1(hh_1)^{-1} f(hh_1 x)\, d(hh_1),$$

$$(6.11) \qquad = \sigma_1(h_1) \int\limits_{H_1} \sigma_1(h')^{-1} f(h'x)\, d(h') = \sigma_1(h_1)(\Pi f)(x).$$

Thus $(\Pi f) \in \sigma_1^X$. Hence we can define $\Delta(f) = \phi(\Pi f) \in V_{\chi_2}$. Then we can check that Δ satisfies the condition (6.7) to be in $\mathcal{D}(X)$. Thus we have an embedding:

$$(6.12) \qquad \mathrm{Hom}_X(\sigma_1^X, \chi_2^X) \hookrightarrow \mathcal{D}(X) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}).$$

- Now we suppose that all spaces $\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_x}, V_{\chi_2}) \simeq \mathrm{Hom}_{({}^x H)}(\sigma_1, \chi_2^x) = 0$ for all orbits y_x but only one orbit y_{x_o}. We consider two cases:

– When the orbit y_{x_o} is open, we have a short exact sequence of distributions:

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y \setminus y_{x_o}), V_{\chi_2}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_{x_o}}, V_{\chi_2}) \longrightarrow 0.$$

In theorem 6.9 of Bernstein and Zelevinsky, in [B,Z], the condition that Y is a Hausdorff space was waived and replaced by its constructibility property, which is sufficient for our argument. Then it shows that the space $\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y \setminus y_{x_o}), V_{\chi_2}) \simeq 0$ since all spaces $\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_x}, V_{\chi_2}) \simeq 0$ for all orbits y_x in Y other than the only open orbit y_{x_o}. Therefore,

$$(6.13) \qquad \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_{x_o}}, V_{\chi_2}).$$

- When the orbit y_{x_o} is not open, we will consider its closure y_c. We have the following short exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(y_c), V_{\chi_2}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y \setminus y_c), V_{\chi_2}) \longrightarrow 0.$$

By the same argument as above, $\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y \setminus y_c), V_{\chi_2}) \simeq 0$, because the spaces $\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_x}, V_{\chi_2}) \simeq 0$ for all orbits y_x in $Y \setminus y_c$. Therefore,

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(y_c), V_{\chi_2}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}). \tag{6.14}$$

The only surviving orbit y_{x_o} is open in its closure y_c. Then we can apply the above case for an open orbit:

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_{y_{x_o}}, V_{\chi_2}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(y_c), V_{\chi_2}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}(Y), V_{\chi_2}). \tag{6.15}$$

Therefore, from the results in (6.10, 6.12) and (6.13, 6.15), we have

$$\mathrm{Hom}_X(\sigma_1^X, \chi_2^X) \hookrightarrow \mathrm{Hom}_{({}^{x_o}H)}(\sigma_1, \chi_2^{x_o}).$$

This completes the proof of Lemma 6.1. ∎

We apply this lemma to the double coset calculation mentioned above.

LEMMA 6.3: *For $i = 0, 1$, let us define ${}^p\mathbf{D}_i = (\mathbf{N}_i.\mathbf{T}_6) \cap (p^{-1}.\mathbf{U}_1.p)$, where $p \in \mathbf{U}_1\backslash\mathbf{P}_6/(\mathbf{N}_i.\mathbf{T}_6)$, $\mathbf{N}_o = \mathbf{G}_4$ and $\mathbf{N}_1 = \mathbf{S}_4$. Define $(\psi.\chi)^p(d) = (\psi.\chi)(p_n.d.p_n^{-1})$, for all d in ${}^p\mathbf{D}_i$. Assume that the space $\mathrm{Hom}_{({}^p\mathbf{D}_i)}(\theta_i, (\psi.\chi)^p) = 0$ for all orbits $\mathbf{U}_1.p.(\mathbf{N}_i.\mathbf{T}_6)$ but only one $\mathbf{U}_1.p_o.(\mathbf{N}_i.\mathbf{T}_6)$. Then for all representations θ_i, where $\theta_1 \in \mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1})$, and $\theta_o \in \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$,*

$$\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},i}(\theta_i), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \hookrightarrow \mathrm{Hom}_{({}^{p_o}\mathbf{D}_i)}(\theta_i, (\psi.\chi)^{p_o}). \tag{6.16}$$

Proof: Same as that of Lemma 6.1, in which $X = \mathbf{P}_6$, $H_1 = (\mathbf{N}_i.\mathbf{T}_6)$, $H_2 = \mathbf{U}_1$, $\sigma_1 = \theta_i$ and $\chi_2 = (\psi.\chi)$. (These groups satisfy conditions in Lemma 6.1.) ∎

In the next section, we will calculate each $\mathrm{Hom}_{({}^p\mathbf{D}_i)}$ space corresponding to each double coset p in $\mathbf{U}_1\backslash\mathbf{P}_6/(\mathbf{N}_i.\mathbf{T}_6)$.

We intend to show that the assumption in the above lemma could be satisfied: for each i, all spaces $\mathrm{Hom}_{({}^p\mathbf{D}_i)}(\theta_i, (\psi.\chi)^p)$ will be trivial, possibly except only one space $\mathrm{Hom}_{({}^{p_o}\mathbf{D}_i)}(\theta_i, (\psi.\chi)^{p_o})$ corresponding to the orbit p_o. (Refer to the chart in Section 0. Introduction.)

7. The double coset calculations for the subgroup $\mathbf{P}_6$

• A. STAGE 1. ◇ a. $i = 1$. The double cosets for this case are in $\mathbf{U}_1\backslash\mathbf{P}_6/(\mathbf{S}_4.\mathbf{T}_6)$.

We have $\mathbf{U}_1.(\mathbf{S}_4.\mathbf{T}_6) = \mathbf{P}_6$, hence there is only one double coset whose representative is p=1. (Its orbit is open.) Let $\theta_1 = \mathcal{J}_-^{\mathbf{L},1}(\pi)$. Then, from Lemma 6.3, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) &\simeq \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},1}(\theta_1), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))\\ &\hookrightarrow \mathrm{Hom}_{\mathbf{D}_1}(\theta_1, (\psi.\chi)^p) = \mathrm{Hom}_{\mathbf{D}_1}(\theta_1, (\psi.\chi)).\end{aligned}$$

Then $f(\theta_1(d)v) = (\psi.\chi)(d)f(v)$, for all $f \in \mathrm{Hom}_{\mathbf{D}_1}(\theta_1, (\psi.\chi))$, $d \in \mathbf{D}_1$ and $v \in V_{\theta_1}$. Let

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & x_5 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$$

in $\mathbf{L}_6$. Then $d \in \mathbf{U}_1 \cap (\mathbf{S}_4.\mathbf{T}_6) = {}^p\mathbf{D}_1 \equiv \mathbf{D}_1$.

– $\theta_1(d) = \Theta_{\mathbf{L},1}(d) = \psi_o(x_5)$, and

– $(\psi.\chi)(d) = \psi\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\chi\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \psi_o(0) = 1.$

Then we can choose x_5 in d, such that $(\psi.\chi)(d) = 1 \neq \theta_1(d)$.

Therefore, the identity $f(\theta_1(d)v) = \Theta_{\mathbf{L},1}(d)f(v) = (\psi.\chi)(d)f(v)$ forces $f(v) = 0$ for all $v \in V_{\theta_1}$. That is, $f = 0$. Thus

$$\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \hookrightarrow \mathrm{Hom}_{\mathbf{D}_1}(\theta_1, (\psi.\chi)) \simeq 0. \tag{7.1}$$

◇ b. $i = 0$. The double cosets for this case are in $\mathbf{U}_1\backslash\mathbf{P}_6/(\mathbf{G}_4.\mathbf{T}_6)$. We have $(\mathbf{G}_4.\mathbf{T}_6) = \mathbf{P}_6$, hence there is only one double coset whose representative is p=1. (Its orbit is open.) Let $\theta_o = \mathcal{J}_-^{\mathbf{L},o}(\pi)$. Then, from Lemma 6.3,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) &\simeq \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},o}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))\\ &\hookrightarrow \mathrm{Hom}_{\mathbf{D}_o}(\theta_o, (\psi.\chi)^p) = \mathrm{Hom}_{\mathbf{D}_o}(\theta_o, (\psi.\chi)).\end{aligned} \tag{7.2}$$

For all $f \in \mathrm{Hom}_{\mathbf{U}_1}(\theta_o, (\psi.\chi))$, $f(\theta_o(d)v) = (\psi.\chi)(d)f(v)$, for all $d \in \mathbf{D}_o$ and $v \in V_{\theta_o}$. We have $\mathbf{D}_o = \mathbf{U}_1 \cap (\mathbf{G}_4.\mathbf{T}_6) = \mathbf{U}_1$. Then the most general form of d

in $\mathbf{U}_1$ is

$$d = \begin{pmatrix} b & x_1 & * & * & * & * \\ & 1 & * & * & * & * \\ & & b & x_1 & * & * \\ & & & 1 & * & * \\ & & & & b & x_1 \\ & & & & & 1 \end{pmatrix}.$$

In general, we cannot compare $(\psi.\chi)(d)$ and $\theta_o(d)$, or, in other words, we cannot claim anything about $f(v)$. Thus the space $\mathrm{Hom}_{\mathbf{U}_1}(\theta_o, (\psi.\chi))$ in (7.2) may not be trivial. We will prove this next.

◇ c. We recall now the short exact sequence at the beginning of Section 6:

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \longrightarrow \mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \longrightarrow \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)).$$

By this sequence, (7.1) and (7.2),

$$\begin{aligned} \mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) &\simeq \mathrm{Hom}_{\mathbf{P}_6}(\mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \\ &\hookrightarrow \mathrm{Hom}_{\mathbf{U}_1}(\theta_o, (\psi.\chi)) \quad (\text{where } \theta_o = \mathcal{J}_-^{\mathbf{L},o}(\pi)) \qquad (7.3) \\ &\simeq \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\theta_o, \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\ &\qquad \text{(by the Frobenius reciprocity theorem)} \\ &\simeq \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)). \qquad (7.4) \end{aligned}$$

The next step is to repeat the work on functors and double coset calculations for the subgroup $(\mathbf{G}_4.\mathbf{T}_6)$.

• B. Stage 2. By (7.4), we reduced our work to investigating only the space

$$\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)).$$

From Proposition I, part 1 proves that all the functors $\mathcal{J}_-^{\mathbf{K},i}$ and $\mathcal{J}_+^{\mathbf{K},i}$ (where $i = 0, 1$) are exact. Therefore, it together with the short exact sequence in part 4 will give us this short exact sequence:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},o}\mathcal{J}_-^{\mathbf{K},o}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\ &\longrightarrow \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\theta_o, \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\ &\longrightarrow \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{K},1}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)). \end{aligned}$$

Eventually, we will prove that $\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},o}\mathcal{J}_-^{\mathbf{K},o}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi))$ is trivial and the space $\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{K},1}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi))$ is at most one-dimensional. For $i=0,1$, let $\sigma_i = \mathcal{J}_-^{\mathbf{K},i}(\theta_o) = \mathcal{J}_-^{\mathbf{K},i}(\mathcal{J}_-^{\mathbf{L},o}(\pi))$. Then we will consider only the following spaces:

$$\begin{aligned}\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}&(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},i}\mathcal{J}_-^{\mathbf{K},i}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi))\\ &\simeq \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},i}(\sigma_i), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)).\end{aligned}$$

In fact, in the following lemma, we would investigate the space

$$\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},i}(\sigma_i), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)),$$

for any representations $\sigma_i \in \mathrm{Alg}(\mathbf{M}_i)$ and $\mathcal{J}_+^{\mathbf{K},i}(\sigma_i)$, which was defined in Section 4.D. We have a lemma which is similar to Lemma 6.3 (same proof).

LEMMA 7.1: *For $i=0,1$, let us define ${}^p\mathbf{D}_i = (\mathbf{M}_i.\mathbf{T}_6)\cap(p^{-1}.\mathbf{U}_1.p)$, where $p \in \mathbf{U}_1\backslash(\mathbf{G}_4.\mathbf{T}_6)/(\mathbf{M}_i.\mathbf{T}_6)$, $\mathbf{M}_o = \mathbf{G}_4$ and $\mathbf{M}_1 = \mathbf{P}_4$. Let $(\psi.\chi)^p(d) = (\psi.\chi)(p.d.p^{-1})$, for all d in ${}^p\mathbf{D}_i$. Assume that the space $\mathrm{Hom}_{({}^p\mathbf{D}_i)}(\Theta_{\mathbf{L},o}.\sigma_i', (\psi.\chi)^p) = 0$ for all orbits $\mathbf{U}_1.p.(\mathbf{M}_i.\mathbf{T}_6)$ but only one $\mathbf{U}_1.p_o.(\mathbf{M}_i.\mathbf{T}_6)$. Then for all representations σ_i in $\mathrm{Alg}(\mathbf{M}_i)$,*

(7.5)
$$\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},i}(\sigma_i), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \hookrightarrow \mathrm{Hom}_{({}^{p_o}\mathbf{D}_i)}(\Theta_{\mathbf{L},o}.\sigma_i', (\psi.\chi)^{p_o}).$$

Using this lemma in the next step, we will calculate each $\mathrm{Hom}_{({}^p\mathbf{D}_i)}$ space corresponding to each double coset p in $\mathbf{U}_1\backslash(\mathbf{G}_4.\mathbf{T}_6)/(\mathbf{M}_i.\mathbf{T}_6)$.

We intend to show that the assumption in the above lemma could be satisfied: for each case i, all spaces $\mathrm{Hom}_{({}^p\mathbf{D}_i)}(\Theta_{\mathbf{L},o}.\sigma_i', (\psi.\chi)^p)$ will be trivial, possibly except only one space $\mathrm{Hom}_{({}^{p_o}\mathbf{D}_i)}(\Theta_{\mathbf{L},o}.\sigma_i', (\psi.\chi)^{p_o})$ which corresponds to the orbit p_o.

◇ a. $i=0$. The double cosets for this case are in $\mathbf{U}_1\backslash(\mathbf{G}_4.\mathbf{T}_6)/(\mathbf{G}_4.\mathbf{T}_6)$; hence there is only one double coset whose representative is p=1. (Its orbit is open.) Let $\sigma_o = \mathcal{J}_-^{\mathbf{K},o}(\theta_o)$. Then, from Lemma 7.1, we have

$$\begin{aligned}\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}&(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},o}\mathcal{J}_-^{\mathbf{K},o}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi))\\ &\hookrightarrow \mathrm{Hom}_{\mathbf{D}_o}(\Theta_{\mathbf{L},o}.\sigma_o', (\psi.\chi)^p) = \mathrm{Hom}_{\mathbf{D}_o}(\Theta_{\mathbf{L},o}.\sigma_o', (\psi.\chi)).\end{aligned}$$

For all $f \in \mathrm{Hom}_{\mathbf{D}_o}(\Theta_{\mathbf{L},o}.\sigma_o', (\psi.\chi))$, we have $f(\Theta_{\mathbf{L},o}.\sigma_o'(d)v) = (\psi.\chi))f(v)$ for all $d \in \mathbf{D}_o$ and $v \in V_{\Theta_{\mathbf{L},o}.\sigma_o'}$. Let

$$d = \begin{pmatrix} 1 & 0 & -x_2 & x_3 & -x_4 & 0 \\ & 1 & & & & x_4 \\ & & 1 & 0 & & x_3 \\ & & & 1 & & x_2 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Then $d \in \mathbf{U}_1 \cap (\mathbf{G}_4.\mathbf{T}_6) = \mathbf{D}_o$. In fact $d \in \mathbf{K}_6$. Let $\mathbf{I}_6$ be the 6×6 identity matrix. Then

- $\Theta_{\mathbf{L},o}.\sigma_o'(d) = \Theta_{\mathbf{K},o}(d).\sigma_o(\mathbf{I}_6) = 1$, and
- $(\psi.\chi)(d) = \psi\begin{pmatrix} 0 & x_3 \\ 0 & x_2 \end{pmatrix}.\chi\begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix} = \psi_o(x_2)$.

Then we can choose x_2 in d, such that $\Theta_{\mathbf{L},o}.\sigma_o'(d) = 1 \neq \psi_o(x_2) = (\psi.\chi)(d)$.

Therefore, the identity $f(\Theta_{\mathbf{L},o}.\sigma_o'(d)v) = f(v) = \psi_o(x_2).f(v) = (\psi.\chi)(d)f(v)$ forces $f(v) = 0$ for all $v \in V_{\Theta_{\mathbf{L},o}.\sigma_o'}$. That is, $f = 0$. Thus

(7.6)

$$\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},o}\mathcal{J}_-^{\mathbf{K},o}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \hookrightarrow \mathrm{Hom}_{\mathbf{D}_o}(\Theta_{\mathbf{L},o}.\sigma_o', (\psi.\chi)) \simeq 0.$$

◇ b. $i = 1$. The double cosets p for this case are in $\mathbf{U}_1\backslash(\mathbf{G}_4.\mathbf{T}_6)/(\mathbf{P}_4.\mathbf{T}_6)$, which is simply equivalent to $\mathbf{U}_1\backslash\mathbf{P}_6/(\mathbf{P}_4.\mathbf{T}_6)$. We now consider the latter double cosets.

Let us define a subgroup $\mathbf{B}_1$ of the Borel subgroup of $\mathbf{GSp}(6, \mathbf{F})$: $\mathbf{B}_1$ consists of matrices of the form

$$\begin{pmatrix} * & * & * & * & * & * \\ & * & * & * & * & * \\ & & * & * & * & * \\ & & & * & * & * \\ & & & & * & * \\ & & & & & 1 \end{pmatrix}.$$

Decomposing $\mathbf{P}_6$, we have

$$\mathbf{P}_6 = \bigcup_{\gamma \in \mathbf{B}_1\backslash\mathbf{P}_6/(\mathbf{P}_4.\mathbf{T}_6);\gamma' \in \mathbf{U}_1\backslash\mathbf{B}_1/[\mathbf{B}_1 \cap (\gamma(\mathbf{P}_6.\mathbf{T}_4)\gamma^{-1})]} \mathbf{U}_1(\gamma'\gamma)(\mathbf{P}_4.\mathbf{T}_6).$$

We can observe $\mathbf{B}_1\backslash\mathbf{P}_6/(\mathbf{P}_4.\mathbf{T}_6) \simeq \mathbf{B}_1'\backslash\mathbf{GSp}(4, \mathbf{F})/\mathbf{P}_4'$, where the Borel subgroup $\mathbf{B}_1'$ consists of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix},$$

and the parabolic subgroup $\mathbf{P}_4'$ consists of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & 1 \end{pmatrix}$$

of the group $\mathbf{GSp}(4,\mathbf{F})$.

By the well-known Bruhat decomposition, we now can identify the Weyl group $\mathcal{W}$ of $\mathbf{GSp}(4,\mathbf{F})$ with $\mathbf{B}_1'\backslash\mathbf{GSp}(4,\mathbf{F})/\mathbf{B}_1'$. It is generated by two permutations of α_1,α_2, and by four transformations of the form $\alpha_1 \to \alpha_1^{\pm 1}$ and $\alpha_2 \to \alpha_2^{\pm 1}$, where α_1,α_2 are non-zero complex numbers. Therefore, the order of the Weyl group $\mathcal{W}$ is eight.

Recalling the generalized Bruhat decomposition (cf. chapter 1.2 in [Ho]), we have $\mathbf{B}_1'\backslash\mathbf{GSp}(4,\mathbf{F})/\mathbf{P}_4' \simeq \mathcal{W}_B\backslash\mathcal{W}/\mathcal{W}_P$, where $\mathcal{W}_B = \mathcal{W}\cap\mathbf{B}_1'$ has only the identity and $\mathcal{W}_P = \mathcal{W}\cap\mathbf{P}_4'$ is a subgroup consisting of the identity and

$$\begin{pmatrix} -1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}.$$

Thus there are four double cosets in $\mathbf{B}_1'\backslash\mathbf{GSp}(4,\mathbf{F})/\mathbf{P}_4'$. Hence we have four double cosets in $\mathbf{B}_1\backslash\mathbf{P}_6/(\mathbf{P}_4.\mathbf{T}_6)$ whose representatives are

$$\gamma_o = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & 1 & & & & \\ & & & -1 & & \\ & & & & & 1 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 1 & & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & -1 & & & \\ & -1 & & & & \\ & & & & & 1 \end{pmatrix} \quad \text{and} \quad \gamma_3 = \begin{pmatrix} 1 & & & & & \\ & & & -1 & & \\ & -1 & & & & \\ & & & & 1 & \\ & & -1 & & & \\ & & & & & 1 \end{pmatrix}.$$

For $j = 0,1,2,3$, we will consider all γ''s associated with each γ_j and recall that p will have the form $p = (\gamma'.\gamma_j)$. Let $\sigma_1 = \mathcal{J}_-^{\mathbf{K},1}(\theta_o)$.

$$\begin{aligned} &\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{K},1}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\ &\qquad \simeq \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},1}(\sigma_1), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)). \end{aligned} \tag{7.7}$$

Then by Lemma 7.1, we will consider the spaces $\mathrm{Hom}_{({}^p\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^p)$ for all p's of the form $p = (\gamma'.\gamma_j)$, where $j = 0,1,2,3$. For all $f \in \mathrm{Hom}_{({}^p\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^p)$, $v \in V_{\Theta_{\mathbf{L},o}.\sigma_1'}$ and $d \in {}^p\mathbf{D}_1$, we have

$$f(\Theta_{\mathbf{L},o}.\sigma_1'(d)v) = (\psi.\chi)^p(d).f(v). \tag{7.8}$$

We now consider four different cases below:

• a. $j = 0$. $\gamma = \gamma_o = 1$. Then $\gamma' \in \mathbf{U}_1\backslash\mathbf{B}_1/[\mathbf{B}_1 \cap (\gamma_o(\mathbf{P}_4.\mathbf{T}_6)\gamma_o^{-1})] = \mathbf{U}_1\backslash\mathbf{B}_1/[\mathbf{B}_1 \cap (\mathbf{P}_4.\mathbf{T}_6)]$. Since $\mathbf{B}_1 \subset (\mathbf{P}_4.\mathbf{T}_6)$, hence $\mathbf{B}_1 \cap (\mathbf{P}_4.\mathbf{T}_6) = \mathbf{B}_1$. That is, $\gamma' = 1$ only; and $p = (\gamma'.\gamma_o) = \gamma_o = 1$. Let

$$d = \begin{pmatrix} 1 & 0 & -x_2 & x_3 & -x_4 & 0 \\ & 1 & & & & x_4 \\ & & 1 & 0 & & x_3 \\ & & & 1 & & x_2 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Then $d \in \mathbf{U}_1 \cap (\mathbf{P}_4.\mathbf{T}_6) = {}^{\gamma_o}\mathbf{D}_1$; in fact, d is in $\mathbf{K}_6$.

- $\Theta_{\mathbf{L},o}.\sigma_1'(d) = \Theta_{\mathbf{K},1}(d).\sigma_1(\mathbf{I}_6) = \psi_o(0) = 1$, and
- $(\psi.\chi)^p(d) = (\psi.\chi)(d) = \psi\begin{pmatrix} 0 & x_3 \\ 0 & x_2 \end{pmatrix}.\chi\begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix} = \psi_o(x_2)$.

Then we can choose x_2 in d, such that $\Theta_{\mathbf{L},o}.\sigma_1'(d) = 1 \neq \psi_o(x_2) = (\psi.\chi)(d)$.

For all $f \in \mathrm{Hom}_{({}^{\gamma_o}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi))$, the identity

$$f(\Theta_{\mathbf{L},o}.\sigma_1'(d)v) = f(v) = \psi_o(x_2)f(v) = (\psi.\chi)(d)f(v)$$

forces $f(v) = 0$ for all $v \in V_{\Theta_{\mathbf{L},o}.\sigma_1'}$. That is, $f = 0$. Thus the space

$$\mathrm{Hom}_{({}^{\gamma_o}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)) \simeq 0. \tag{7.9}$$

• b. $j = 1$: $\gamma = \gamma_1 = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & 1 & & & & \\ & & & -1 & & \\ & & & & & 1 \end{pmatrix}$.

To consider $\gamma' \in \mathbf{U}_1\backslash\mathbf{B}_1/[\mathbf{B}_1 \cap (\gamma_1(\mathbf{P}_4.\mathbf{T}_6)\gamma_1^{-1})]$, we observe that

$$\mathbf{Y}_1 \stackrel{\text{def}}{=} \mathbf{B}_1 \cap [\gamma_1(\mathbf{P}_4.\mathbf{T}_6)\gamma_1^{-1}] = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ & * & * & & * & * \\ & & 1 & & & * \\ & & & * & * & * \\ & & & & * & * \\ & & & & & 1 \end{pmatrix} \in \mathbf{GSp}(6,\mathbf{F}) \right\}.$$

Since $(\mathbf{U}_1.\mathbf{Y}_1) = \mathbf{B}_1$, we have only one $\gamma' = 1$ and hence $p = (\gamma'.\gamma_1) = \gamma_1$. Let

$$d = \begin{pmatrix} 1 & 0 & -x_2 & x_3 & -x_4 & 0 \\ & 1 & & & & x_4 \\ & & 1 & 0 & & x_3 \\ & & & 1 & & x_2 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Then

$$d' = \begin{pmatrix} 1 & x_3 & 0 & x_4 & -x_2 & 0 \\ & 1 & & & & x_2 \\ & & 1 & & & x_4 \\ & & & 1 & & 0 \\ & & & & 1 & x_3 \\ & & & & & 1 \end{pmatrix}$$

will satisfy the identity $d.\gamma_1 = \gamma_1.d'$, and $d' \in (\gamma_1^{-1}.\mathbf{U}_1.\gamma_1) \cap (\mathbf{P}_4.\mathbf{T}_6) = {}^{\gamma_1}\mathbf{D}_1$; in fact, both d and d' are in $\mathbf{K}_6$. We have:

- $\Theta_{\mathbf{L},o}.\sigma_1'(d') = \Theta_{\mathbf{K},1}(d').\sigma_1(\mathbf{I}_6) = \psi_o(x_3)$, and
- $(\psi.\chi)^{\gamma_1}(d') = (\psi.\chi)(\gamma_1.d'.\gamma_1^{-1}) = (\psi.\chi)(d) = \psi\begin{pmatrix} 0 & x_3 \\ 0 & x_2 \end{pmatrix}.\chi\begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix}$
$= \psi_o(x_2)$.

Then we can choose x_2 and x_3 in d such that $\Theta_{\mathbf{L},o}.\sigma_1'(d') = \psi_o(x_3) \neq \psi_o(x_2) = (\psi.\chi)^{\gamma_1}(d')$. For all $f \in \mathrm{Hom}_{({}^{\gamma_1}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^{\gamma_1})$, the identity

$$f(\Theta_{\mathbf{L},o}.\sigma_1'(d')v) = \psi_o(x_3).f(v) = \psi_o(x_2).f(v) = (\psi.\chi)^{\gamma_1}(d')f(v)$$

forces $f(v) = 0$ for all $v \in V_{\Theta_{\mathbf{L},o}.\sigma_1'}$. That is, $f = 0$. Thus the space

(7.10) $$\mathrm{Hom}_{({}^{\gamma_1}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^{\gamma_1}) \simeq 0.$$

• c. $j = 2$. $\gamma = \gamma_2 = \begin{pmatrix} 1 & & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & -1 & & & \\ & -1 & & & & \\ & & & & & 1 \end{pmatrix}.$

We have a similar result:

(7.11) $$\mathrm{Hom}_{({}^{\gamma_2}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^{\gamma_2}) \simeq 0.$$

• d. $j = 3$. $\gamma = \gamma_3 = \begin{pmatrix} 1 & & & & & \\ & & & -1 & & \\ & -1 & & & & \\ & & & & 1 & \\ & & -1 & & & \\ & & & & & 1 \end{pmatrix}$.

Let us consider $\gamma' \in \mathbf{U}_1\backslash\mathbf{B}_1/[\mathbf{B}_1 \cap (\gamma_3(\mathbf{P}_4.\mathbf{T}_6)\gamma_3^{-1})]$, which is a set of diagonal matrices of the form: $\mathrm{diag}(z, z, 1, z, 1, 1)$, for all $z \in \mathbf{F}^\times$. Therefore,

$$p = (\gamma'.\gamma_3) = \begin{pmatrix} z & & & & & \\ & & & -z & & \\ & -1 & & & & \\ & & & & z & \\ & & -1 & & & \\ & & & & & 1 \end{pmatrix}, \quad \text{for all } z \in \mathbf{F}^\times. \tag{7.12}$$

Suppose $d' \in (\mathbf{P}_4.\mathbf{T}_6)$. To satisfy the condition $d = p.d'.p^{-1} \in \mathbf{U}_1$, the most general form of d' is

$$d' = \begin{pmatrix} a & x_1' & x_2' & x_3 & x_4' & x_5 \\ & a & y' & & z.x_3 & x_4 \\ & & a & & & x_3 \\ & & b & 1 & y & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix}. \tag{7.13}$$

(Then $d' \in {}^p\mathbf{D}_1$.)

- $\Theta_{\mathbf{L},o}.\sigma_1'(d') = \psi_o(x_1).\sigma_1 \begin{pmatrix} a & & & & & \\ & a & y' & & z.x_3 & \\ & & a & & & \\ & & b & 1 & y & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$, and

- $(\psi.\chi)^p(d') = (\psi.\chi)(p.d'.p^{-1}) = (\psi.\chi)(d) = \psi\begin{pmatrix} y' & -x_4 \\ 0 & z.x_1 \end{pmatrix}.\chi\begin{pmatrix} a & -x_3 \\ & 1 \end{pmatrix}$

$$= \psi_o(y' + z.x_1).\chi\begin{pmatrix} a & -x_3 \\ & 1 \end{pmatrix} = \psi_o(y' + z.x_1).|a|^s.$$

Recalling that, for all $f \in \mathrm{Hom}_{({}^p\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^p)$, $d' \in {}^p\mathbf{D}_1$ and $v \in V_{\Theta_{\mathbf{L},o}.\sigma_1'}$, we have the identity

$$f(\Theta_{\mathbf{L},o}.\sigma_1'(d')v) = (\psi.\chi)^p(d')f(v). \tag{7.8}$$

For each $z \neq 1$ fixed, we have two cases:

- If $\psi_o(z) \neq \psi_o(1)$, let $a = 1, x_3 = 0, y = y' = 0, b = 0$ and $x_1 = 1$ in d'. Then

$$\Theta_{\mathbf{L},o}.\sigma_1'(d') = \psi_o(1) \neq \psi_o(z) = (\psi.\chi)^p(d').$$

Therefore, (7.8) gives $\psi_o(1).f(v) = \psi_o(z).f(v)$. This identity forces $f(v) = 0$ for all $v \in V_{\Theta_{\mathbf{L},o}.\sigma_1'}$. That is, $f = 0$.

- If $\psi_o(z) = \psi_o(1)$, let $x_3 = 0, y = y' = 0, b = 0$ and $x_1 = 0$ in d'. Let $\mathbf{I}_a$ be a diagonal matrix of the form $(a, a, a, 1, 1, 1)$. Then

$$\Theta_{\mathbf{L},o}.\sigma_1'(d') = \sigma_1(\mathbf{I}_a) \quad \text{and} \quad (\psi.\chi)^p(d') = |a|^s, \quad \text{for almost all } s.$$

Then, (7.8) is rewritten as

$$f(\sigma_1(\mathbf{I}_a)v) = |a|^s.f(v).$$

This equation has solutions for at most a finite number of s in the complex plane. That is, $f = 0$, for almost all s. Thus, for all matrices p described in (7.12) above (where $z \neq 1$) and for almost all s,

$$\text{(7.14)} \qquad \mathrm{Hom}_{({}^p\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^p) \simeq 0.$$

Finally, we consider only one case when $z = 1$ (hence $p = \gamma_3$). Generally, we cannot compare $\Theta_{\mathbf{L},o}.\sigma_1'(d)$ and $(\psi.\chi)^{\gamma_3}(d)$ for $d \in {}^{\gamma_3}\mathbf{D}_1$. In other words, the space $\mathrm{Hom}_{({}^{\gamma_3}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^{\gamma_3})$ may not be trivial, for any representation σ_1 in $\mathrm{Alg}(\mathbf{P}_4)$.

Now, we will collect results in (7.7) and (7.9), (7.10), (7.11) and (7.14). By Lemma 7.1 we can conclude, for almost all s,

(7.15)

$$\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{K},1}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \hookrightarrow \mathrm{Hom}_{({}^{\gamma_3}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^p).$$

• C. We recall now the short exact sequence preceding Lemma 7.1:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},o}\mathcal{J}_-^{\mathbf{K},o}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\ &\longrightarrow \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\theta_o, \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\ &\longrightarrow \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{K},1}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)). \end{aligned}$$

From this sequence, (7.6) and (7.15), we have

$$\begin{aligned} &\mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\theta_o, \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\ &\qquad \underset{\dim}{\leq} \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_+^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{K},1}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)). \end{aligned}$$

For any two spaces A and B, by the notation: $A \underset{\dim}{\leq} B$, we mean the dimension of A is not greater than that of B: $\dim(A) \leq \dim(B)$. Then, from (7.4) and (7.15),

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \\
(7.4) \quad &\hookrightarrow \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_{-}^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\
&\underset{\dim}{\leq} \mathrm{Hom}_{(\mathbf{G}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\mathcal{J}_{+}^{\mathbf{K},o}\mathcal{J}_{-}^{\mathbf{K},o}(\theta_o), \mathrm{ind}_{\mathbf{U}_1}^{(\mathbf{G}_4.\mathbf{T}_6)}(\psi.\chi)) \\
&\qquad\qquad \text{(by the above sequence and (7.6))} \\
(7.16) \quad &\hookrightarrow \mathrm{Hom}_{({}^{\gamma_3}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^{\gamma_3}) \quad \text{(by (7.15)) for almost all } s.
\end{aligned}$$

In the next stage, we will prove that $\mathrm{Hom}_{({}^{\gamma_3}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^{\gamma_3})$ may not be trivial.

Let $\mathbf{H}_1 = {}^{\gamma_3}\mathbf{D}_1 = (\mathbf{P}_4.\mathbf{T}_6) \cap (\gamma_3^{-1}.\mathbf{U}_1.\gamma_3)$. Let $\tau = (\psi.\chi)^{\gamma_3}$. That is, $\tau(h_1) = (\psi.\chi)(\gamma_3.h_1.\gamma_3^{-1})$ for all $h_1 \in \mathbf{H}_1$. Recall $\mathcal{J}_{-}^{\mathbf{L},o}(\pi) = \theta_o$, $\mathcal{J}_{-}^{\mathbf{K},1}(\theta_o) = \sigma_1$. Then

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \\
(7.17) \quad &\underset{\dim}{\leq} \mathrm{Hom}_{({}^{\gamma_3}\mathbf{D}_1)}(\Theta_{\mathbf{L},o}.\sigma_1', (\psi.\chi)^{\gamma_3}) \\
&\simeq \mathrm{Hom}_{(\mathbf{P}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\sigma_1', \mathrm{ind}_{\mathbf{H}_1}^{(\mathbf{P}_4.\mathbf{T}_6)}\tau) \\
&\qquad\qquad \text{(by the Frobenius reciprocity theorem)} \\
&\simeq \mathrm{Hom}_{(\mathbf{P}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\mathcal{J}_{-}^{\mathbf{K},1}\mathcal{J}_{-}^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{H}_1}^{(\mathbf{P}_4.\mathbf{T}_6)}\tau), \\
(7.18) \quad &\qquad\qquad \text{for almost all } s.
\end{aligned}$$

Then the next step is to repeat the work on functors and double coset calculations for the subgroup $(\mathbf{P}_4.\mathbf{T}_6)$, in which the subgroup $\mathbf{H}_1$ will play the role of $\mathbf{U}_1$ in the previous sections (particularly, in Lemmas 6.3 and 7.1). Explicitly,

$$\begin{aligned}
\mathbf{H}_1 &= (\gamma_3^{-1}.\mathbf{U}_1.\gamma_3) \cap (\mathbf{P}_4.\mathbf{T}_6) \\
&= \left\{ h_1 \in \mathbf{GSp}(6,\mathbf{F}) \,\middle|\, h_1 = \begin{pmatrix} a & x_1' & x_2' & z & x_4' & x_5 \\ & a & y_1' & & z & x_4 \\ & & a & & & z \\ & & w & 1 & y_1 & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix} \right\}.
\end{aligned}$$

Then

$$\gamma_3.\mathbf{H}_1.\gamma_3^{-1} = \left\{ \begin{pmatrix} a & -z & -x_1' & x_4' & -x_2' & x_5 \\ & 1 & & -y_1 & w & -x_2 \\ & & a & -z & y_1' & -x_4 \\ & & & 1 & & x_1 \\ & & & & a & -z \\ & & & & & 1 \end{pmatrix} \in \mathbf{GSp}(6,\mathbf{F}) \right\},$$

and

$$\tau(h_1) = (\psi.\chi)(\gamma_3.h_1.\gamma_3^{-1}) = \psi\begin{pmatrix} y_1' & -x_4 \\ & x_1 \end{pmatrix}.\chi\begin{pmatrix} a & -z \\ & 1 \end{pmatrix}$$

$$(7.19) \qquad = \psi_o(x_1 + y_1').\chi\begin{pmatrix} a & -z \\ & 1 \end{pmatrix}, \text{ for all } h_1 \in \mathbf{H}_1.$$

8. The calculations on the subgroup $(\mathbf{P}_4.\mathbf{T}_6)$ and $(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)$

Repeating some calculations done in Sections 4–7, we will get similar results for the subgroups $\mathbf{P}_4$ and $(\mathbf{P}_4.\mathbf{T}_6)$.

$$\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi))$$

$$(7.18) \quad \underset{\dim}{\leq} \mathrm{Hom}_{(\mathbf{P}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{H}_1}^{(\mathbf{P}_4.\mathbf{T}_6)}\tau)$$

$$(8.1) \quad \simeq \mathrm{Hom}_{(\mathbf{G}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\mathcal{J}_-^{\mathbf{L},o}(\sigma_1)), \mathrm{ind}_{\mathbf{H}_1}^{(\mathbf{G}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau)$$

$$(8.2) \quad \underset{\dim}{\leq} \mathrm{Hom}_{\mathbf{H}_2}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\eta_1', \tau^*), \qquad \text{for almost all } s,$$

where $\mathbf{H}_2 = (\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6) \cap \mathbf{H}_1$, $\sigma_1 = \mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}(\pi)$ and $\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}(\sigma_1) = \eta_1$. Then

$$\mathrm{Hom}_{\mathbf{H}_2}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\eta_1', \tau)$$

$$\simeq \mathrm{Hom}_{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\eta_1', \mathrm{ind}_{\mathbf{H}_2}^{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau_2)$$

(by the Frobenius reciprocity theorem)

$$(8.3) \quad \simeq \mathrm{Hom}_{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\eta_1, \mathrm{ind}_{\mathbf{H}_2}^{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau_2)$$

$$(8.4) \quad \simeq \mathrm{Hom}_{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}(\pi), \mathrm{ind}_{\mathbf{H}_2}^{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau_2)$$

It is true for almost all s.

The next step is to repeat the work on functors and double coset calculations for the subgroup $(\mathbf{P}_2.\mathbf{T}_4) = \mathbf{T}_6$, in which the subgroup $\mathbf{H}_2$ will play the role of $\mathbf{H}_1$.

Explicitly, from the definition of $\mathbf{H}_1$ and τ in (7.19),

$$\mathbf{H}_2 = (\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6) \cap \mathbf{H}_1$$

$$= \left\{ h_2 \middle| \; h_2 = \begin{pmatrix} a & x_1' & x_2' & z & x_4' & x_5 \\ & a & y_1' & & z & x_4 \\ & & a & & & z \\ & & & 1 & y_1 & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix} \in \mathbf{GSp}(6, \mathbf{F}) \right\},$$

and for all $h_2 \in \mathbf{H}_2$,

$$\tau_2(h_2) = \tau(h_2) = \psi \begin{pmatrix} y_1' & -x_4 \\ & x_1 \end{pmatrix} . \chi \begin{pmatrix} a & -z \\ & 1 \end{pmatrix} = \psi_o(x_1 + y_1') . \chi \begin{pmatrix} a & -z \\ & 1 \end{pmatrix}. \tag{8.5}$$

We will have similar results for the subgroups $\mathbf{P}_2$ and $(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)$. We recall $\mathcal{J}_-^{\mathbf{L},o}(\pi) = \theta_o$, $\mathcal{J}_-^{\mathbf{K},1}(\theta_o) = \sigma_1$, $\mathcal{J}_-^{\mathbf{L},o}(\sigma_1) = \lambda_o$, $\mathcal{J}_-^{\mathbf{K},1}(\lambda_o) = \eta_1$, and $\mathcal{J}_-^1(\eta_1) = \xi_1$. So far, we have proved

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}_{\mathbf{U}_1}^{\mathbf{P}_6}(\psi.\chi)) \\ &\underset{\dim}{\leq} \mathrm{Hom}_{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}(\pi), \\ &\qquad \mathrm{ind}_{\mathbf{H}_2}^{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau_2) \qquad \text{(by (8.4))} \\ &\underset{\dim}{\leq} \mathrm{Hom}_{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\mathcal{J}_+^1\mathcal{J}_-^1(\eta_1), \mathrm{ind}_{\mathbf{H}_2}^{(\mathbf{P}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau_2) \\ &\hookrightarrow \mathrm{Hom}_{\mathbf{H}_3}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\xi_1', \tau_2), \end{aligned} \tag{8.6}$$

where $\mathbf{H}_3 = (\mathbf{P}_o.\mathbf{T}_2.\mathbf{T}_4.\mathbf{T}_6) \cap \mathbf{H}_2$. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{H}_3}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\xi_1', \tau_2) \\ &\simeq \mathrm{Hom}_{(\mathbf{P}_o.\mathbf{T}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\Omega_1.\xi_1), \mathrm{ind}_{\mathbf{H}_3}^{(\mathbf{P}_o.\mathbf{T}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau_3) \\ &\simeq \mathrm{Hom}_{(\mathbf{P}_o.\mathbf{T}_2.\mathbf{T}_4.\mathbf{T}_6)}(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\Omega_1.\mathcal{J}_-^1\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{K},1}\mathcal{J}_-^{\mathbf{L},o}(\pi), \\ &\qquad \mathrm{ind}_{\mathbf{H}_3}^{(\mathbf{P}_o.\mathbf{T}_2.\mathbf{T}_4.\mathbf{T}_6)}\tau_3), \end{aligned} \tag{8.7}$$

for almost all s. By the definition of $\mathbf{H}_2$ and τ_2 in (8.5), we have

$$\mathbf{H}_3 = (\mathbf{P}_o.\mathbf{T}_2.\mathbf{T}_4.\mathbf{T}_6) \cap \mathbf{H}_2 = \left\{ h_3 \middle| \, h_3 = \begin{pmatrix} 1 & x_1 & x_2' & z & x_4' & x_5 \\ & 1 & y_1 & & z & x_4 \\ & & 1 & & & z \\ & & & 1 & y_1 & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix} \in \mathbf{GSp}(6, \mathbf{F}) \right\}.$$

Let us denote its character $\tau_3 = \tau_2$. Then for all $h_3 \in \mathbf{H}_3$,

$$\tau_3(h_3) = \psi \begin{pmatrix} -y_1' & -x_4 \\ & x_1 \end{pmatrix} . \chi \begin{pmatrix} 1 & -z \\ & 1 \end{pmatrix} = \psi_o(x_1 + y_1) . \chi \begin{pmatrix} 1 & -z \\ & 1 \end{pmatrix}. \tag{8.8}$$

We can observe $(\mathbf{P}_o.\mathbf{T}_2.\mathbf{T}_4.\mathbf{T}_6) = \mathbf{N}$, which is the maximal unipotent subgroup of $\mathbf{GSp}(6,\mathbf{F})$ consisting of matrices of the form

$$n = \begin{pmatrix} 1 & x_1 & * & * & * & * \\ & 1 & y_1 & * & * & * \\ & & 1 & u & * & * \\ & & & 1 & y_1 & * \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix},$$

and

$$(\Theta_{\mathbf{L},o}.\Theta_{\mathbf{K},1}.\Lambda_{\mathbf{L},o}.\Lambda_{\mathbf{K},1}.\Omega_1)(n) = \psi_o(x_1+y_1+u) \overset{\text{def}}{=} \Psi_{\mathbf{N}}(n).$$

From the definitions of $\mathcal{J}^1_-, \mathcal{J}^{\mathbf{K},1}_-, \mathcal{J}^{\mathbf{L},o}_-, \mathcal{J}^{\mathbf{K},1}_-, \mathcal{J}^{\mathbf{L},o}_-$ which were used in Sections 7 and 8, we can observe that $\mathcal{J}^1_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-(\pi)$ is a representation of the subgroup $\mathbf{P}_o = 1$, hence it is simply a trivial representation. Thus, from (8.7),

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}^{\mathbf{P}_6}_{\mathbf{U}_1}(\psi.\chi)) \\ &\underset{\dim}{\leq} \mathrm{Hom}_{\mathbf{N}}(\Psi_{\mathbf{N}}.\mathcal{J}^1_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-(\pi), \mathrm{ind}^{\mathbf{N}}_{\mathbf{H}_3}\tau_3) \\ (8.9)\quad &\simeq \mathrm{Hom}_{\mathbf{H}_3}(\Psi_{\mathbf{N}}.\mathcal{J}^1_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-(\pi), \tau_3), \qquad \text{for almost all } s, \end{aligned}$$

where τ_3, defined in (8.8), is a character of $\mathbf{H}_3$, since χ is just a character of $\mathbf{B}'_2$ which is the subgroup of $\mathbf{B}_2$, consisting of matrices of the form $b_1 = \begin{pmatrix} * & * \\ & 1 \end{pmatrix}$ in $\mathbf{GL}(2,\mathbf{F})$. (Recall Section 2.) Because of the uniqueness of the Whittaker model, the space $\mathrm{Hom}_{\mathbf{H}_3}(\Psi_{\mathbf{N}}.\mathcal{J}^1_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-\mathcal{J}^{\mathbf{K},1}_-\mathcal{J}^{\mathbf{L},o}_-(\pi), \tau_3)$ is at most one-dimensional.

Thus, for almost all s in the complex plane, the space $\mathrm{Hom}_{\mathbf{P}_6}(\pi, \mathrm{ind}^{\mathbf{P}_6}_{\mathbf{U}_1}(\psi.\chi))$ is at most one-dimensional. This conclusion will complete the proof of statement (2.5) and of Theorem 1, when we finish proving Proposition I (stated in Section 5).

9. The proof of Proposition I

The symplectic group $\mathbf{GSp}(6,\mathbf{F})$ is a closed subgroup of the group $\mathbf{GL}(6,\mathbf{F})$ which is totally disconnected and locally compact. Therefore, $\mathbf{GSp}(6,\mathbf{F})$ itself is also a totally disconnected and locally compact group.

We will be able to use some results stated and proved in chapter I of [B,Z]. Here, we demonstrate only the proof of part 4 of the proposition:

For all representations $\theta_o \in \mathrm{Alg}(\mathbf{G}_4.\mathbf{T}_6, \Theta_{\mathbf{L},o})$, $\theta_1 \in \mathrm{Alg}(\mathbf{S}_4.\mathbf{T}_6, \Theta_{\mathbf{L},1})$, *and* $\pi \in \mathrm{Alg}(\mathbf{P}_6)$, *let us consider the homomorphisms:*

$$\ell_o\colon \pi \longrightarrow \mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi); \quad \ell_o'\colon \mathcal{J}_-^{\mathbf{L},o}\mathcal{J}_+^{\mathbf{L},o}(\theta_o) \longrightarrow \theta_o,$$
$$\ell_1\colon \mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi) \longrightarrow \pi; \quad \ell_1'\colon \theta_1 \longrightarrow \mathcal{J}_-^{\mathbf{L},1}\mathcal{J}_+^{\mathbf{L},1}(\theta_1).$$

Then ℓ_o' *and* ℓ_1' *are isomorphisms, and* ℓ_o *and* ℓ_1 *form an exact short sequence:*

$$(9.1) \qquad 0 \longrightarrow \mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi) \xrightarrow{\ell_1} \pi \xrightarrow{\ell_o} \mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi) \longrightarrow 0.$$

Proof: The homomorphisms ℓ_o and ℓ_1 are well-defined. Indeed, ℓ_1 corresponds to the identity $(\mathbf{S}_4.\mathbf{T}_6)$-homomorphism $\mathcal{J}_-^{\mathbf{L},1}(\pi) \to \mathcal{J}_-^{\mathbf{L},1}(\pi)$ and ℓ_1' corresponds to the identity $\mathbf{P}_6$-homomorphism $\mathcal{J}_+^{\mathbf{L},1}(\theta_1) \to \mathcal{J}_+^{\mathbf{L},1}(\theta_1)$ in (5.1). Similarly, ℓ_o and ℓ_o' correspond to the identity $(\mathbf{G}_4.\mathbf{T}_6)$-homomorphisms $\mathcal{J}_-^{\mathbf{L},o}(\pi) \to \mathcal{J}_-^{\mathbf{L},o}(\pi)$ and the identity $\mathbf{P}_6$-homomorphism $\mathcal{J}_+^{\mathbf{L},o}(\theta_o) \to \mathcal{J}_+^{\mathbf{L},o}(\theta_o)$ in (5.2). And ℓ_o' and ℓ_1' are isomorphisms.

Now, the main problem is to prove that sequence (9.1) is exact.

◇ Let $g(A)$ denote an element in $\mathbf{G}_4$ of the form $\begin{pmatrix} a & & \\ & [A] & \\ & & 1 \end{pmatrix}$, where the matrix $[A]_{4\times 4}$ is an element in $\mathbf{GSp}(4,\mathbf{F})$. Obviously, $\mu(g(A)) = \mu([A]) = a \in \mathbf{F}^\times$, where $\mu(g)$ is the similitude factor of g. (Then $g(A')$ is in $\mathbf{S}_4$ if $\mu([A']) = 1$.)

Let $l(y)$ be an element in $\mathbf{L}_6$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & y \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Then

$$\Theta_{\mathbf{L},o}(l(y)) = 1 \quad \text{and} \quad \Theta_{\mathbf{L},1}(l(y)) = \psi_o(y).$$

We have

$$(9.2) \qquad g(A^{-1}).l(y).g(A) = l(\mu([A])^{-1}.y).$$

We recall that $\mathbf{L}_6$ is the central subgroup of $\mathbf{T}_6$. Therefore, the identity (9.2) can be rewritten as

$$(9.3) \qquad [g(A).t]^{-1}.l(y).[g(A).t] = l(\mu([A])^{-1}.y) = l(a^{-1}.y),$$

for all $[g(A).t] \in (\mathbf{G}_4.\mathbf{T}_6)$.

$\diamond$ We have $\mathbf{L}_6 \simeq \mathbf{F}$, which is a totally disconnected and locally compact topological space. Then, we can define the sheaves on $\mathbf{F}$ by using only a base $\mathcal{B}_{oc}$ of open compact subsets.

Let $(\pi, \mathbf{P}_6, V_\pi)$ be a smooth representation. We make V_π a $C_c^\infty(\mathbf{F})$-module via Fourier transform $\hat{f}(y) = \int_{\mathbf{F}} f(x)\psi_o(xy)\,dx$ by

$$f{\bullet}v = \pi(\hat{f})v, \qquad \text{where } \pi(f)v = \int_{\mathbf{F}} f(y)\pi(l(y))v\,dy. \tag{9.4}$$

For all f_1, f_2 in $C_c^\infty(\mathbf{F})$, v in V_π, we have

$$f_1{\bullet}v = \int_{\mathbf{F}}\int_{\mathbf{F}} f_1(x)\psi_o(xy)\pi(l(y))v\,dx\,dy.$$

Then $(f_1.f_2){\bullet}v = f_1{\bullet}(f_2{\bullet}v)$. Therefore, V_π is a $C_c^\infty(\mathbf{F})$-module.

Since V_π is a smooth $\mathbf{P}_6$-module, hence it is also a *cosmooth* $C_c^\infty(\mathbf{F})$-module, i.e. for any v in V_π, there exists some open compact subset K of $\mathbf{F}$ such that $1_K{\bullet}v = v$, where 1_K is some characteristic function of K. Then the action can also be extended to make V_π a $C^\infty(\mathbf{F})$-module via π', which is defined by

$$\pi'(f)v = \pi(f.1_K)v, \qquad \text{for all } f \text{ in } C^\infty(\mathbf{F}). \tag{9.5}$$

$\diamond$ We want to construct the sheaf $\mathcal{F}$ of these modules.

For all open compact subsets U of $\mathbf{F}$, let $\tilde{\mathcal{F}}(U) = \{v \in V_\pi | 1_U{\bullet}v = v\}$. If $W \subseteq U$, we define a restriction map $\rho_{U,W}\colon \tilde{\mathcal{F}}(U) \to \tilde{\mathcal{F}}(W)$ by $\rho_{U,W}(v) = 1_W{\bullet}v$. Then $\tilde{\mathcal{F}}$ is a presheaf.

Let $\mathcal{F}$ be the sheaf associated with the presheaf $\tilde{\mathcal{F}}$. Then $\mathcal{F}(U) = \tilde{\mathcal{F}}(U)$ when U is an open compact subset.

Now, we use the result of exercise 1.19, chapter II in [H], where the closed subspace $Z = \{0\}$ and the open subspace $Y = \mathbf{F} \setminus Z = \mathbf{F}^\times$. Let $\mathcal{F}'$ be the sheaf associated with the presheaf $\tilde{\mathcal{F}}'$, where

$$\tilde{\mathcal{F}}'(U) = \begin{cases} \mathcal{F}(U), & \text{if } 0 \notin U, \\ 0, & \text{if } 0 \in U. \end{cases}$$

We also define the skyscraper sheaf $\mathcal{F}''$ as

$$\mathcal{F}''(U) = \begin{cases} \mathcal{F}_0, & \text{if } \quad 0 \in U, \\ 0, & \text{if } \quad 0 \notin U. \end{cases}$$

Their stalks can be expressed as

$$(9.6)\quad \mathcal{F}'_x = \tilde{\mathcal{F}}'_x = \mathcal{F}_x, \quad \mathcal{F}''_x = 0 \quad \text{if } x \neq 0 \quad \text{and} \quad \mathcal{F}'_0 = \tilde{\mathcal{F}}'_0 = 0, \quad \mathcal{F}''_0 = \mathcal{F}_0.$$

Then we have the short exact sequence

$$(9.7)\qquad 0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

- ◇ If a is in $\mathbf{F}^\times$, we can consider a topological automorphism (in fact, a homeomorphism) α: $\mathbf{F}^\times \longrightarrow \mathrm{Aut}(\mathbf{F})$, defined as: $\alpha(a)(x) = ax$ for all x in $\mathbf{F}$.

Now we consider the action of $g(A)$ on $\mathcal{F}$. For all f in $C_c^\infty(\mathbf{F})$,

$$\begin{aligned} f \bullet \pi(g(A))v &= \int_{\mathbf{F}}\int_{\mathbf{F}} f(x)\psi_o(xy)\pi(l(y))\pi(g(A))v \, dx\, dy \\ &= \int_{\mathbf{F}}\int_{\mathbf{F}} f(x)\psi_o(xy)\pi(g(A))\pi(l(a^{-1}y))v \, dx\, dy \quad \text{(by the identity (9.2))}. \end{aligned}$$

We may make a change of variables $y \to ay$, $x \to a^{-1}x$:

$$f \bullet \pi(g(A))v = \pi(g(A)) \int_{\mathbf{F}}\int_{\mathbf{F}} f(a^{-1}x)\psi_o(xy)\pi(l(y))v \, dx\, dy.$$

Let us define $f_a(x) = f(ax)$. Then we can write this result as

$$f \bullet \pi(g(A))v = \pi(g(A))(f_{a^{-1}} \bullet v) \quad \text{for all } v \text{ in } V_\pi.$$

This expression is independent of matrix $[A]$ in $g(A)$. Therefore, it makes sense to denote $\Pi_a \equiv \pi(g(A))$. Then we may rewrite that expression as

$$(9.8)\qquad f_a \bullet \Pi_a v = \Pi_a(f \bullet v) \quad \text{for all } v \text{ in } V_\pi.$$

We will now show that this expression implies that $\mathcal{F}'$ is isomorphic to a constant sheaf $\mathcal{F}_1$ on $\mathbf{F}^\times$ extended by zero to a sheaf on $\mathbf{F}$. Indeed, let us observe

$$f_a(x) = 1 \iff ax = \alpha(a)(x) \in U \iff x \in a^{-1}U \text{ or } x \in \alpha(a^{-1})U.$$

Then $f = 1_U \iff f_a = 1_{\alpha(a^{-1})U}$. Using these functions in (9.8), we have

$$\Pi_a v = \Pi_a(1_U . v) = 1_{\alpha(a^{-1})U}\Pi_a v, \quad \text{for all } v \text{ in } \mathcal{F}(U).$$

Therefore, $\Pi_a v \in \mathcal{F}(\alpha(a^{-1})U)$. In other words, $\Pi_{a^{-1}}$ induces an isomorphism between $\mathcal{F}(U) \leftrightarrow \mathcal{F}(\alpha(a)U)$.

On stalks, it also induces an isomorphism (denoted the same) which acts transitively on the stalks:

$$\Pi_{a^{-1}}\colon \mathcal{F}_x \longrightarrow \mathcal{F}_{ax} \quad \text{for all } x \text{ in } \mathbf{F}^\times. \tag{9.9}$$

Therefore, all the stalks are isomorphic. Then by (9.6), $\mathcal{F}'_x = \mathcal{F}_x \simeq \mathcal{F}_{ax} = \mathcal{F}'_{ax}$.

$\diamond$ We will prove that $\mathcal{F}'$ and $\mathcal{F}''$ are flasque sheaves.

First, we check that the sheaf $\mathcal{F}$ is flasque. That is, when $W \subsetneq U$, we prove that the restriction map $\rho_{U,W}\colon \mathcal{F}(U) \to \mathcal{F}(W)$ defined by $\rho_{U,W}(v) = 1_W \bullet v$ is surjective.

It is obvious, since for all v in $\mathcal{F}(W)$, we have $1_W \bullet v = v$, so $1_U \bullet v = (1_U.1_W)\bullet v = 1_W \bullet v = v$. Then v is in $\mathcal{F}(U)$, too. Therefore, the restriction map $\rho_{U,W}(v) = 1_W \bullet v = v$ is surjective. Thus $\mathcal{F}$ is a flasque sheaf. Then we can embed $\mathcal{F}(W) \subset \mathcal{F}(U)$ as subspaces of V_π.

From the definition of $\mathcal{F}'$ in (9.6), we have $\mathcal{F}'(U) = \mathcal{F}(U)$ for all open compact subsets $U \not\ni 0$. In general,

$$\mathcal{F}'(U) = \\ \Big\{\text{sections } S \text{ in } \mathcal{F}(U), S\colon U \to \bigcup_{a\in U} \mathcal{F}_a \text{ such that } S \text{ vanishes at } 0 \text{ if } 0 \in U\Big\}.$$

Then $S \in \mathcal{F}'(U)$ implies $S \in \mathcal{F}'(U')$ for some $U' \subset U$ and $0 \notin U'$. But $\mathcal{F}'(U') \subset \mathcal{F}'(U)$, therefore, $\mathcal{F}'(U) = \mathcal{F}'(U')$. Hence we may assume $0 \notin U$. Then $\mathcal{F}'(U) = \mathcal{F}(U)$ for all U. Therefore, $\mathcal{F}'$ is flasque.

In the exact sequence (9.7), both $\mathcal{F}$ and $\mathcal{F}'$ are flasque sheaves, hence $\mathcal{F}''$ is also a flasque sheaf (exercise 1.16, chapter II, in [H]). Now we need some lemmas.

LEMMA 9.1:. *The following are equivalent:*

- $v \in V(\mathbf{L}_6, \Theta_{\mathbf{L},o})$,
- $1_U \bullet v = 0$, *for some open compact subset U containing* 0.

Proof: The Fourier transform of $1_{\wp^n}$ is const $.1_{\wp^{-n}}$. Then

$$
\begin{aligned}
& 1_U \bullet v = 0, \text{ for some open compact subset } U \text{ containing } 0, \\
\Longleftrightarrow\ & 1_{\wp^n} \bullet v = 0, \text{ for some } n \iff \text{const}\,.1_{\wp^{-n}} \bullet v = 0 \\
\Longleftrightarrow\ & \int_{\wp^{-n}} \pi(l(y)) v\, dy. \quad \text{We may make a change of variable } n \to -n \\
\Longleftrightarrow\ & \int_{\wp^n} \pi(l(y)) v\, dy = \int_{\wp^n} \Theta_{\mathbf{L},o}(l(y)) \pi(l(y)) v\, dy \\
\Longleftrightarrow\ & v \in V(\mathbf{L}_6, \Theta_{\mathbf{L},o})
\end{aligned}
$$

(by a lemma of Jacquet–Langlands, stated in section 2.33 of [B,Z]). ∎

LEMMA 9.2: *The Stalk at 0:* $\mathcal{F}_0 = V_\pi / V(\mathbf{L}_6, \Theta_{\mathbf{L},o})$, *where* $V(\mathbf{L}_6, \Theta_{\mathbf{L},o}) = \langle (\pi(l)v - v)$, *for all* v *in* V_π *and* l *in* $\mathbf{L}_6 \rangle$. *That is,*

$$\mathcal{F}_0 = \mathcal{J}_-^{\mathbf{L},o}(\pi) = \mathcal{J}_+^{\mathbf{L},o} \mathcal{J}_-^{\mathbf{L},o}(\pi). \tag{9.10}$$

Proof: By definition, $\mathcal{F}_0$ is the direct limit of the groups $\mathcal{F}(U)$ for all open compact subsets U containing 0, via the restriction maps. We denote by $\mathcal{F}_0 = \varinjlim_{U \ni 0} \mathcal{F}(U)$.

To prove this limit is $V_\pi / V(\mathbf{L}_6, \Theta_o)$, we need to check two conditions:

[1] Define $\tau_U \colon \mathcal{F}(U) \longrightarrow V_\pi / V(\mathbf{L}_6, \Theta_{\mathbf{L},o})$ by $\tau_U =$ Proj $\circ$ Incl, where Incl is an inclusion mapping and Proj is a projection mapping defined naturally as

$$\mathcal{F}(U) \xrightarrow{\text{Incl}} V_\pi \xrightarrow{\text{Proj}} V_\pi / V(\mathbf{L}_6, \Theta_{\mathbf{L},o}).$$

For all v in $\mathcal{F}(U)$, and $W \subset U$, we must check $\tau_U(v) = \tau_W \circ \rho_{U,W}(v)$. Indeed,

$$1_W \bullet (v - \rho_{U,W}(v)) = 1_W \bullet v - 1_W^2 \bullet v = 0.$$

Therefore, by Lemma 9.1, we have $(v - \rho_{U,W}(v)) \in V(\mathbf{L}_6, \Theta_{\mathbf{L},o})$. Thus $\tau_U(v - \rho_{U,W}(v)) = 0$. In other words,

$$\tau_U(v) = \tau_U(\rho_{U,W}(v)) = \tau_W(\rho_{U,W}(v)).$$

[2] Next, we check this condition:

If there exists a vector space T and η_U: $\mathcal{F}(U) \longrightarrow T$ are vector space homomorphisms for all open compact subsets U such that this diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho_{U,W}} & \mathcal{F}(W) \\ & {\scriptstyle \eta_U} \searrow \quad \swarrow {\scriptstyle \eta_W} & \\ & T & \end{array}$$

is commutative. Then there exists uniquely a homomorphism η: $\mathcal{F}_0 \longrightarrow T$ such that for all U, we have $\eta_U = \eta \circ \tau_U$.

Indeed, given v in both $\mathcal{F}(U)$ and $\mathcal{F}(W)$, where U, W both contain 0, it suffices to prove that $\eta_U(v) = \eta_W(v)$. Indeed, we consider the restriction map

$$\rho_{U\cup W,U}\colon \mathcal{F}(U \cup W) \longrightarrow \mathcal{F}(U).$$

It is not isomorphic, but if $v \in \mathcal{F}(U)$ then $v \in \mathcal{F}(U \cup W)$ and $\rho_{U\cup W,U}(v) = v$. Therefore, this commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U \cup W) & \xrightarrow{\rho_{U\cup W,U}} & \mathcal{F}(U) \\ & {\scriptstyle \eta_{U\cup W}} \searrow \quad \swarrow {\scriptstyle \eta_U} & \\ & T & \end{array}$$

implies that

$$\eta_U(v) = \eta_U \circ \rho_{U\cup W,W}(v) = \eta_{U\cup W}(v).$$

Then, by symmetry,

$$\eta_U(v) = \eta_W(v)(= \eta_{U\cup W}(v)).$$

THE UNIQUENESS: If there exists another homomorphism η': $\mathcal{F}_0 \longrightarrow T$ such that, for all U, we have $\eta_U = \eta' \circ \tau_U$, then we will prove $\eta = \eta'$. For all v in $\mathcal{F}_0$, there exists w_U in $\mathcal{F}(U)$ such that $\tau_U(w_U) = v$. Hence

$$\eta'(v) = \eta' \circ \tau_U(w_U) = \eta_U(w_U) = \eta \circ \tau_U(w_U) = \eta(v).$$

Thus $\eta' = \eta$. ∎

LEMMA 9.3: *The Stalk at 1: $\mathcal{F}_1 = V_\pi / V(\mathbf{L}_6, \Theta_{\mathbf{L},1})$ where*

$$V(\mathbf{L}_6, \Theta_{\mathbf{L},1}) = \langle (\pi(l)v - \Theta_{\mathbf{L},1}(l)v), \text{ for all } v \text{ in } V_\pi \text{ and } l \text{ in } \mathbf{L}_6 \rangle .$$

That is,

(9.11) $\mathcal{F}_1 = \mathcal{J}_{-}^{\mathbf{L},1}(\pi)$ *is an* $(\mathbf{S}_4.\mathbf{T}_6)$*-module.*

Proof: By definition, $\mathcal{F}_1$ is the direct limit of the groups $\mathcal{F}(U)$ for all open compact subsets U containing 1, via the restriction maps. We denote $\mathcal{F}_1 = \lim_{\rightarrow U\ni 1} \mathcal{F}(U)$. The proof is similar to that of Lemma 9.2, except that we will use the following lemma in place of Lemma 9.1. ∎

LEMMA 9.4: *The following are equivalent:*

- $v \in V(\mathbf{L}_6, \Theta_{\mathbf{L},1})$,
- $1_U \bullet v = 0$, *for some open compact subset U containing* 1.

Proof: Let F_n be the Fourier transform of the characteristic function $1_{(1+\wp^n)}$. Then

$$F_n(x) = \int_{(1+\wp^n)} \psi_o(xy)\,dy = \int_{\wp^n} \psi_o(x(1+y))\,dy$$
$$= \psi_o(x) \int_{\wp^n} \psi_o(xy)\,dy = \psi_o(x) \int_{\mathbf{F}} 1_{\wp^n}(y)\psi_o(xy)\,dy.$$

Therefore, $F_n(x) = \psi_o(x).1_{\wp^{-n}}/\mathrm{vol}(\wp^{-n})$. Then

$$1_U \bullet v = 0, \text{ for some open compact subset } U \text{ containing } 1$$
$$\Longleftrightarrow 1_{(1+\wp^n)}.v = 0 \text{ for some } n \text{ large}$$
$$\Longleftrightarrow F_n.v = 0 \text{ for some } n \text{ large}$$
$$\Longleftrightarrow (1/\mathrm{vol}(\wp^{-n})) \int_{\mathbf{F}} 1_{\wp^{-n}}(y)\psi_o(y)\pi(l(y))v\,dy = 0.$$

We may make a change of variable $n \to -n$,

$$\Longleftrightarrow (1/\mathrm{vol}(\wp^{n})) \int_{\mathbf{F}} 1_{\wp^{n}}(y)\psi_o(y)\pi(l(y))v\,dy = 0$$
$$\Longleftrightarrow \int_{\wp^n} \psi_o(y).\pi(l(y))v\,dy = 0$$
$$\Longleftrightarrow \int_{\wp^n} \Theta_{\mathbf{L},1}(l(y))\pi(l(y))v\,dy$$

$\Longleftrightarrow v \in V(\mathbf{L}_6, \Theta_{\mathbf{L},1})$ (by the lemma of Jacquet–Langlands, loc. cit.). ∎

◇ Now we define the space

$$\mathcal{F}'_c(U) = \Big\{\text{all locally constant, compactly supported sections } S\colon U \longrightarrow \bigcup_{a\in U} \mathcal{F}_a\Big\}$$

for all compact open subsets $U \subset \mathbf{F}^\times$. We could extend any section S to a function

$$S'\colon \mathbf{P}_6(U) \longrightarrow \bigcup_{a\in U} \mathcal{F}_a, \qquad \text{by } S'(g) = S(\mu(g)) = S(a),$$

where $\mu(g) = a$, for some a in $U \subset \mathbf{F}^\times$ and $\mathbf{P}_6(U) = \{g \in \mathbf{P}_6 | \mu(g) \in U\}$. Then, obviously, $\mathbf{P}_6(\mathbf{F}^\times)$ is just $\mathbf{P}_6$.

By this definition, if $\mu(g_1) = \mu(g_2)$ then $S'(g_1) = S'(g_2)$.

(Conversely, given function S', for all a in $\mathbf{F}^\times$, there exists some $g = g(A)$ in $\mathbf{P}_6$, such that

$$\mu(g(A)) = \mu(A) = a.$$

Let us define $S''(a) = S'((g(A))$. Then $S'' = S'$.)

In the case when $U = \mathbf{F}^\times$, the following lemma holds.

LEMMA 9.5:

$$\mathcal{F}'_c(\mathbf{F}^\times) = \mathcal{J}_+^{\mathbf{L},1}(\mathcal{F}_1). \tag{9.12}$$

Proof: Let $\theta = \mathcal{J}_-^{\mathbf{L},1}(\pi)$. We will consider this representation $(\theta, (\mathbf{S}_4.\mathbf{K}_6), \mathcal{F}_1)$. Then $\mathcal{J}_+^{\mathbf{L},1}(\mathcal{F}_1)$ is the space of all locally constant and compactly supported functions $\varphi\colon \mathbf{P}_6 \longrightarrow \mathcal{F}_1$ such that

$$\varphi(s.t.p) = \theta(s.t).\varphi(p), \quad \text{for all } (s.t) \in (\mathbf{S}_4.\mathbf{T}_6) \text{ and } p \in \mathbf{P}_6. \tag{9.13}$$

Particularly,

$$\varphi(l.p) = \theta(l).\varphi(p) = \Theta_{\mathbf{L},1}(l).\varphi(p) = \psi_o(y).\varphi(p),$$

by assuming $l = l(y)$ in $\mathbf{L}_6$. From (9.9), we can observe an isomorphism between these two spaces:

$$-\ \varphi(g) \overset{\text{def}}{=} \pi(g)S'(g) = \Pi_a S'(g) = \Pi_a S(a), \quad \text{where } a = \mu(g). \tag{9.14}$$

$$-\ S(a) = S'(g(A)) \overset{\text{def}}{=} \pi(g(A)^{-1})\varphi(g(A)) = \Pi_{a^{-1}}\varphi(g(A)), \tag{9.15}$$

where $g = g(A)$ is in $\mathbf{P}_6$, such that $\mu(g(A)) = a$, and we define $\Pi_{a^{-1}} = \pi(g(A)^{-1})$, such that the obvious conditions

$$\Pi_a \circ \Pi_{a^{-1}} = \mathrm{id}_{\mathcal{F}_1} \quad \text{and} \quad \Pi_{a^{-1}} \circ \Pi_a = \mathrm{id}_{\mathcal{F}_a}$$

could be satisfied (where $\Pi_a \colon \mathcal{F}_a \longrightarrow \mathcal{F}_1$, $\Pi_{a^{-1}} \colon \mathcal{F}_1 \longrightarrow \mathcal{F}_a$). We need to check the compatibility of Π_a with the group action on these two subspaces $\mathcal{F}_a$ and $\mathcal{F}_1$.

First, we can generalize Lemma 9.3 to get the stalk $\mathcal{F}_a$, for some $a \in \mathbf{F}^\times$:

$$\mathcal{F}_a \stackrel{\text{def}}{=} \lim_{\substack{\rightarrow \\ U \ni a}} \mathcal{F}(U) = V_\pi / V(\mathbf{L}_6, \Theta_{\mathbf{L},1}, a),$$

where

$$V(\mathbf{L}_6, \Theta_{\mathbf{L},1}, a) = \langle (\pi(l(y))v - \Theta_{\mathbf{L},1}(l(a.y))v), \text{ for all } v \text{ in } V_\pi, l \text{ in } \mathbf{L}_6 \rangle .$$

Then $V(\mathbf{L}_6, \Theta_{\mathbf{L},1}, 1) \equiv V(\mathbf{L}_6, \Theta_{\mathbf{L},1})$. For all $w \in V(\mathbf{L}_6, \Theta_{\mathbf{L},1}, a)$,

$$\begin{aligned} \Pi_a(w) &= \pi(g(A))[\pi(l(y))v - \Theta_{\mathbf{L},1}(l(a.y))v] \qquad \text{for some } v \in V_\pi,\ l(y) \in \mathbf{L}_6, \\ &= \pi(l(a.y))\pi(g(A))v - \Theta_{\mathbf{L},1}(l(a.y)).\pi(g(A))v) \qquad \text{(by (9.3))}. \end{aligned}$$

Let $l'(y) = l(a.y)$. When y runs through $\mathbf{F}^\times$, $l'(y)$ runs through $\mathbf{L}_6$. Let $v' = \pi(g(A))v$. Then

$$\Pi_a(w) = [\pi(l'(y))v' - \Theta_{\mathbf{L},1}(l'(y))v'] \in V(\mathbf{L}_6, \Theta_{\mathbf{L},1}.$$

Thus it makes sense to write

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(\mathbf{L}_6, \Theta_{\mathbf{L},1}, a) & \longrightarrow & V_\pi & \longrightarrow & \mathcal{F}_a & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Pi_a} & & \downarrow{\scriptstyle \Pi_a} & & \downarrow{\scriptstyle \Pi_a} & & \\ 0 & \longrightarrow & V(\mathbf{L}_6, \Theta_{\mathbf{L},1}) & \longrightarrow & V_\pi & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0. \end{array}$$

Now, we need to check two conditions:

[1] For all $(s.t) \in (\mathbf{S}_4.\mathbf{T}_6)$ and $p \in \mathbf{P}_6$,

$$\begin{aligned} \varphi(s.t.p) &\stackrel{\text{def}}{=} \pi(s.t.p)S'(s.t.p) = \pi(s.t)\pi(p)S(\mu(s.t.p)) = \pi(s.t)\pi(p)S(\mu(p)) \\ &= \pi(s.t)\varphi(p) = \theta(s.t)\varphi(p) = \mathcal{J}_{-}^{\mathbf{L},1}(\pi)(s.t)\varphi(p) \quad (\text{since } \mu(s.t) = 1). \end{aligned}$$

The function φ, which is defined in (9.14), satisfies (9.13).

Next, we check that $S(a)$, which is defined in (9.15), is a section in $\mathcal{F}'_c(U)$. We have $\varphi(g(A)) \in \mathcal{F}_1$, hence $S(a) \stackrel{\text{def}}{=} \Pi_{a^{-1}}\varphi(g(A))$ is in $\mathcal{F}_a$.

Let some set $\{U_i\} \subset \mathcal{B}_{oc}$ be a cover of U. For all U_i, we consider the section $S_i \in \mathcal{F}'_c(\mathbf{U}_i)$ where

$$\mathcal{F}'_c(\mathbf{U}_i) = \Big\{\text{locally constant and compactly supported sections } S\colon U_i \longrightarrow \bigcup_{a\in U} \mathcal{F}_a\Big\}.$$

Then $S_i(a) = S(a)$ for all $a \in U_i$. Therefore, $\rho_{U_i,a}(S_i) = S_i(a) = S(a)$ for all U_i's.

[2] The smoothness of V_π implies these two equivalent conditions:

$$S \text{ is locally constant section} \iff \varphi \text{ is locally constant function.}$$

All the above arguments allow us to conclude $\mathcal{J}_+^{\mathbf{L},1}(\mathcal{F}_1) = \mathcal{F}'_c(\mathbf{F}^\times)$. ∎

Therefore, this lemma and Lemma 9.3 will give us $\mathcal{F}'_c(\mathbf{F}^\times) = \mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi)$. By extending $\mathcal{F}'_c(\mathbf{F}^\times)$ trivially to 0, we have

$$\mathcal{F}'_c(\mathbf{F}) = \mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi). \tag{9.16}$$

Let

$$\mathcal{F}''_c(U) = \begin{cases} \mathcal{F}_0, & \text{if } 0 \in U, \\ 0, & \text{if } 0 \notin U. \end{cases}$$

By Lemma 11.5, we have

$$\mathcal{F}''_c(\mathbf{F}) = \mathcal{F}_0 = \mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi). \tag{9.17}$$

We define $\mathcal{F}_c(U') = \bigcup_{U\subset U'} \tilde{\mathcal{F}}(U)$, where the union is over open compact subsets U. Then

$$\mathcal{F}_c(\mathbf{F}) = \bigcup_{U\subset\mathbf{F}} \tilde{\mathcal{F}}(U) = \bigcup_{U\subset\mathbf{F}} \mathcal{F}(U).$$

By the cosmoothness of V_π, for any v in V_π, there exists some open compact subset K of $\mathbf{F}$, such that $1_K\bullet v = v$, or $v \in \tilde{\mathcal{F}}(K) = \mathcal{F}(K)$. Therefore,

$$\bigcup_{U\subset\mathbf{F}} \tilde{\mathcal{F}}(U) = V_\pi = \mathcal{F}_c(\mathbf{F}). \tag{9.18}$$

From (9.7), the short sequence $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is exact. We proved $\mathcal{F}'$ is flasque, hence by exercise 1.16, chapter II in [H], the sequence

$$0 \longrightarrow \mathcal{F}'(\mathbf{F}) \xrightarrow{h_1} \mathcal{F}(\mathbf{F}) \xrightarrow{h_2} \mathcal{F}''(\mathbf{F}) \longrightarrow 0 \tag{9.19}$$

is exact, too. Let $h'_1 = h_1|_{\mathcal{F}'_c(\mathbf{F})}$ and $h'_2 = h_2|_{\mathcal{F}_c(\mathbf{F})}$. Then the next lemma holds.

LEMMA 9.6: *The following short sequence is exact:*

$$0 \longrightarrow \mathcal{F}'_c(\mathbf{F}) \xrightarrow{h'_1} \mathcal{F}_c(\mathbf{F}) \xrightarrow{h'_2} \mathcal{F}''_c(\mathbf{F}) \longrightarrow 0. \tag{9.20}$$

Proof: We need to check some conditions:
- $h_1\colon \mathcal{F}'(\mathbf{F}) \hookrightarrow \mathcal{F}(\mathbf{F})$ is injective. $\mathrm{Ker}(h'_1) \subset \mathrm{Ker}(h_1) = 0$. Thus $\mathrm{Ker}(h'_1) = 0$. Hence $h'_1\colon \mathcal{F}'_c(\mathbf{F}) \hookrightarrow \mathcal{F}_c(\mathbf{F})$ is injective.
- The exactness in (9.19) gives us $h_2(\mathcal{F}'(\mathbf{F})) = 0$. By the definition of h'_2, we have $h'_2(\mathcal{F}'_c(\mathbf{F})) = h_2(\mathcal{F}'_c(\mathbf{F})) \subset h_2(\mathcal{F}'(\mathbf{F})) = 0$. Therefore, $h'_2(\mathcal{F}'_c(\mathbf{F})) = 0$.
- The last condition: h'_2 is surjective. For all T in $\mathcal{F}''_c(\mathbf{F}) = \mathcal{F}''(\mathbf{F})$, since h_2 is surjective, there exists an S in $\mathcal{F}(\mathbf{F})$ such that $h_2(S) = T$. We can choose S to be compactly supported. Indeed, consider some section S_1 in $\mathcal{F}'_c(\mathbf{F})$; then $(S.S_1)$ is a compactly supported section in $\mathcal{F}_c(\mathbf{F})$. By the definition of h'_2,

$$h'_2(S.S_1) = h_2(S.S_1) = h_2(S) + h_2(S_1) = h_2(S) + h'_2(S_1) = T + 0 = T. \quad \blacksquare$$

Finally, from (9.16), (9.17), (9.18) and (9.20), we can write

$$0 \longrightarrow \mathcal{J}_+^{\mathbf{L},1}\mathcal{J}_-^{\mathbf{L},1}(\pi) \longrightarrow \pi \longrightarrow \mathcal{J}_+^{\mathbf{L},o}\mathcal{J}_-^{\mathbf{L},o}(\pi) \longrightarrow 0.$$

10. The local functional equation

It will be helpful to modify our notations slightly to emphasize local calculations in this section and for the rest of this paper.

Let $\mathbf{F}$ be the global field, and $\mathbf{F}_v$ be a non-archimedean local field equipped with its ring of integers $\mathcal{O}_v$. The residue field has the order $N_v = q$.

Let $\mathbf{B}_{2,v} = \mathbf{B}_2(\mathbf{F}_v)$ be the Borel subgroup of $\mathbf{GL}_{2,v} = \mathbf{GL}(2, \mathbf{F}_v)$. Let $(\rho_s, \mathbf{GL}_{2,v}, V_{\rho_s}) = \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s$ and $(\rho_{1-s}, \mathbf{GL}_{2,v}, V_{\rho_{1-s}}) = \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^{1-s}$.

Let $(\pi_v, \mathbf{GSp}(6, \mathbf{F}_v), V_{\pi_v})$ be an irreducible, smooth and generic cuspidal representation as in Theorem 1. We identify the space V_{π_v} with its Whittaker model $\mathcal{W}_{\pi_v}$.

We now rephrase Theorem 1 in a simpler case.

COROLLARY 10.1: *For almost all s in the complex plane, there exists at most one non-trivial $\mathbf{GL}_{2,v}$-invariant bilinear form on $V_{\pi_v} \times V_{\rho_s}$, up to a constant multiple.*

In other words, the space $\mathrm{Hom}_{\mathbf{GL}_{2,v}}(\pi_v, \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s)$ *is at most one-dimensional, for almost all s in the complex plane.*

Proof: Let

$$\mathbf{i}(g) = \begin{pmatrix} g & & \\ & g & \\ & & g \end{pmatrix}$$

describe an embedding $\mathbf{i}\colon \mathbf{GL}_{2,v} \to \mathbf{GSp}(6, \mathbf{F}_v)$. Then the bilinear form B in (1.1) satisfies

$$B[\pi(\mathbf{i}(g))v, \rho(g)w] = B(v,w), \qquad \text{for all } v \in V_{\pi_v}, w \in V_{\rho_s} \text{ and } g \in \mathbf{GL}_{2,v}.$$

Thus B is invariant under the action of $\mathbf{GL}_{2,v}$. Theorem 1 gives the uniqueness of this bilinear form on $V_{\pi_v} \times V_{\rho_s}$. ∎

Let $\zeta_{\mathbf{F}}$ be the Dedekind zeta function of the global field $\mathbf{F}$. Then $\zeta_{\mathbf{F}} = \prod_v \zeta_{\mathbf{F}_v}$, where $\zeta_{\mathbf{F}_v}(s) = (1 - N_v^{-s})^{-1} = (1 - q_v^{-s})^{-1}$ at the non-archimedean places and are normalized gamma functions at the archimedean places.

We recall the definition of the integral $Z_v(s, W_v, f_{s,v})$, where $W_v \in \mathcal{W}_{\pi_v}$ and $f_{s,v} \in V_{\rho_s}$, in [B,G]:

$$Z_v(s, W_v, f_{s,v}) = \zeta_{\mathbf{F}_v}(2s) \int\limits_{\mathbf{B}_{2,v}\backslash\mathbf{GL}_{2,v}} \int\limits_{\mathbf{F}_v^\times} \int\limits_{\mathbf{F}_v^2} W_v\left(\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & u & z & 1 & & \\ & & -u & & 1 & \\ & & & & & 1 \end{pmatrix} .\gamma^{-1}.\mathbf{i}(g)\right) \times |a|^{s-3}.f_{s,v}(g)\, dz\, du\, d^\times a\, dg, \tag{10.1}$$

where $\gamma = \gamma_3$ (defined in Section 7). Then $Z_v(s, W_v, f_{s,v})$ is convergent for sufficiently large $\Re(s)$.

Proposition 10.2: *Let $\mathbf{F}_v$ be a non-archimedean local field whose residue field is of cardinality q. Then the integral $Z_v(s, W_v, f_{s,v})$ defines a rational function of variable q^{-s}. Hence, particularly, $Z_v(s, W_v, f_{s,v})$ has a meromorphic continuation to all s.*

Proof: Let $\check{Z}_v(s, W_v, f_{s,v}) = Z_v(s, W_v, f_{s,v})/\zeta_{\mathbf{F}_v}(2s)$. Then, obviously, we need to prove only that $\check{Z}_v(s, W_v, f_{s,v})$ is a rational function of q^{-s}.

Let Y denote the tensor product space $V_{\pi_v} \otimes \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s$. Then the action Γ_s of $\mathbf{GL}_{2,v}$ on Y can be described as

$$\Gamma_s(g)y = \Gamma_s(g)(W_v \otimes f_{s,v}) = \pi(\mathbf{i}(g))W_v \otimes \rho_s(g)(f_{s,v}). \tag{10.2}$$

Thus the uniqueness of the bilinear form on $V_{\pi_v} \times \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s$ in Corollary 10.1 can be translated into the uniqueness of the linear form on $Y = V_{\pi_v} \otimes \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s$. Let $T_s(y) = T_s(W_v \otimes f_{s,v}) \stackrel{\mathrm{def}}{=} \check{Z}_v(s, W_v, f_{s,v})$. Then we can check that

$$T_s(\Gamma_s(g)y) = \check{Z}_v(s, \pi(\mathbf{i}(g))W_v, \rho_s(g)f_{s,v}) = \check{Z}_v(s, W_v, f_{s,v}) = T_s(y)$$

by Proposition 10.4, which will be proved later.

There exist some function $f_{s,v}^o$ and Whittaker function W_v^o satisfying

$$\check{Z}_v(s, W_v^o, f_{s,v}^o) = 1. \tag{10.2a}$$

That is, $T_s(y^o) = T_s(W_v^o \otimes f_{s,v}^o) = 1$. Therefore, for all but finitely many s (or q^{-s}) there is a unique non-trivial linear functional $T_s \colon Y \to \mathbb{C}$ such that

$$T_s(\Gamma_s(g)y - y) = 0 \quad \text{and} \quad T_s(W_v^o \otimes f_{s,v}^o) = 1. \tag{10.3}$$

From this setup, we now can follow the ideas used to prove Proposition 10.3 in [G,PS].

Let D be a multiplicative subgroup of $\mathbb{C}$ which is regarded as an irreducible algebraic variety over $\mathbb{C}$. By parametrizing z in D by $z = q^{-s}$, we have the ring of polynomials $\mathbb{C}[D] = \mathbb{C}[z, z^{-1}] = \mathbb{C}[q^s, q^{-s}]$.

Let D_1 be the subset of D such that (10.3) has a unique solution for all q^{-s} in D_1. (Hence D_1 is nonempty and open.)

We now view the equations in (10.3) as a family of systems of equations for the dual space $Y^* = \mathrm{Hom}_{\mathbb{C}}(Y, \mathbb{C})$ by considering the collection Ξ_s of pairs $\{(\Gamma_s(g_i)y_j - y_j, 0); (y^o, 1)\}$ indexed by some countable index set $\mathcal{I} = \{(i,j)\}$, for all q^{-s} in D_1 (since Y has a countable dimension over $\mathbb{C}$). Then the system Ξ_s has the unique solution T_s in Y^* for each q^{-s} in D_1.

By definition in [Be], the family $\{\Xi_s\}$ is polynomial in q^{-s} since all systems Ξ_s are indexed by the same set $\mathcal{I}$ and since, for each (i,j), $(\Gamma_s(g_i)y_j - y_j)$ is in Y, which is embedded in $Y \otimes \mathbb{C}[q^s, q^{-s}]$. Hence it is polynomial in q^{-s}.

Thus Bernstein's theorem (in [Be]) implies the existence of a linear functional $T\colon Y\otimes L \longrightarrow L$, where L is the field of fractions of $\mathbb{C}[D]$, and $T(y\otimes q^{-s}) = T_s(y)$ for all y in Y, and all q^{-s} in D_1. Therefore, T_s is a rational function of variable q^{-s} and so are $\check{Z}_v(s, W_v, f_{s,v})$ and $Z_v(s, W_v, f_{s,v})$. ∎

The immediate result of Corollary 10.1 is local functional equation.

Let $M_{s,v}$: $\mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^{s} \to \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^{1-s}$ be a normalized intertwining operator defined as

$$(M_{s,v}f_{s,v})(g) = \int_{\mathbf{F}_v} f_{s,v}\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} .g\right) dx$$

for all $f_{s,v}$ in V_{ρ_s}. This proves that $V_{\rho_s} \simeq V_{\rho_{1-s}}$ (Jacquet–Langlands' theorem).

PROPOSITION 10.3 (The local functional equation): *Assume that $\mathbf{F}_v$ is a non-archimedean local field whose residue cardinality is q. Then there exists a meromorphic function $\gamma_v(s)$ such that, for almost all s,*

$$Z_v(s, W_v, f_{s,v}) = \gamma_v(s).Z_v(1-s, W_v, M_{s,v}f_{s,v}). \tag{10.4}$$

In fact, $\gamma_v(s)$ is a rational function of q^{-s}.

Proof: Let us denote $\tilde{f}_{1-s,v} = M_{s,v}f_{s,v}$. Since both integrals $Z_v(s, W_v, f_{s,v})$ and $Z_v(1-s, W_v, \tilde{f}_{1-s,v})$ are $\mathbf{GL}_{2,v}$-invariant bilinear forms on $V_{\pi_v}\times V_{\rho_s} \simeq V_{\pi_v}\times V_{\rho_{1-s}}$, Corollary 10.1 asserts that there exists a factor $\gamma_v = \gamma_v(s)$ such that (10.4) could be satisfied.

It is a meromorphic function and, moreover, a rational function of q^{-s}, because the integral Z_v is also, by Proposition 10.2 above. ∎

• By (10.2a) in the proof of Proposition 10.2, it remains to prove the following:

PROPOSITION 10.4: *For any non-archimedean local place v, there exist a Whittaker function $W_v^o \in \mathcal{W}_{\pi_v}$ and a function $f_{s,v}^o \in \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^{s}$ such that*

$$\check{Z}_v(s, W_v^o, f_{s,v}^o) \equiv 1.$$

Proof: We denote $\mathbf{K}_v = \mathbf{GL}(2, \mathcal{O}_v)$, the maximum compact subgroup of $\mathbf{GL}_{2,v}$, and observe that $\mathbf{B}_{2,v}\backslash\mathbf{GL}_{2,v} \cap \mathbf{K}_v)\backslash\mathbf{K}_v$. Then we can rewrite the integral:

$$\check{Z}_v(s, W_v, f_{s,v}) = Z_v(s, W_v, f_{s,v})/\zeta_{\mathbf{F}_v}(2s)$$

$$= \int_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v} \int_{\mathbf{F}_v^\times} \int_{\mathbf{F}_v^2} W_v\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.\gamma^{-1}.\mathbf{i}(\tilde{k})\right)$$

$$\times\ |a|^{s-3}.f_{s,v}(\tilde{k})dz\,du\,d^\times a\,d\tilde{k}. \tag{10.5}$$

• STEP 1: Let $\mathcal{S}(\mathbf{F}_v)$ be the set of all Schwartz functions ν on $\mathbf{F}_v$, $\nu\colon \mathbf{F}_v \longrightarrow \mathbb{C}$.

There is some Whittaker function W_1 in $\mathcal{W}_{\pi_v}$ such that $W_1(1) \neq 0$, and then it can be normalized to $W_1(1) = 1$.

• STEP 2: Let us define a function $A_1(a) = W_1(\mathrm{diag}(a,a,a,1,1,1))$. Then $A_1(1) = 1$. Suppose a Schwartz function ν_1 in $\mathcal{S}(\mathbf{F}_v)$ is chosen such that its Fourier transform, defined by $\hat{\nu}_1(a) = \int_{\mathbf{F}_v} \nu_1(x).\psi_o(a.x)\,dx$ (which is also a Schwartz function in $\mathcal{S}(\mathbf{F}_v)$), is supported on a sufficiently small neighborhood of 1 in $\mathcal{O}_v^\times$, and $\int_{\mathcal{O}_v^\times} \hat{\nu}_1(a).A_1(a)\,d^\times a = 1$.

We define a Whittaker function W_2 in $\mathcal{W}_{\pi_v}$ by

$$W_2(g) = \int_{\mathbf{F}_v} \nu_1(x).(\rho\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & x && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} W_1)(g)\,dx. \tag{10.6}$$

Then

$$\int_{\mathbf{F}_v^\times} W_2(\mathrm{diag}(a,a,a,1,1,1)).|a|^{s-3}\,d^\times a$$

$$= \int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v} \nu_1(x).W_1\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & x && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\right)$$

$$\times\ |a|^{s-3}\,dx\,d^\times a$$

$$= \int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v} \nu_1(x).W_1\left(\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 & a.x && \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\right)$$

$$\times\, |a|^{s-3}\, dx\, d^{\times}a$$

$$= \int_{\mathbf{F}_v^{\times}} \int_{\mathbf{F}_v} \nu_1(x).\psi_o(a.x).W_1(\mathrm{diag}(a,a,a,1,1,1)).|a|^{s-3}\, d^{\times}a$$

$$(10.7)\quad = \int_{\mathbf{F}_v^{\times}} \hat{\nu}_1(a).A_1(a).|a|^{s-3}\, d^{\times}a = \int_{\mathcal{O}_v^{\times}} \hat{\nu}_1(a).A_1(a)\, d^{\times}a = 1.$$

• STEP 3: Let

$$A_2(z) = \int_{\mathbf{F}_v^{\times}} W_2 \begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & & z & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} .|a|^{s-3}\, d^{\times}a.$$

Then by (10.7) we have $A_2(0) = 1$. Suppose that $\nu_2\colon \mathbf{F}_v \longrightarrow \mathbb{C}$ is a Schwartz function in $\mathcal{S}(\mathbf{F}_v)$ which is chosen such that its Fourier transform, defined by $\hat{\nu}_2(z) = \int_{\mathbf{F}_v} \nu_2(x).\psi_o(z.x)\, dx$, is supported on a sufficiently small neighborhood of 0, and $\int_{\mathbf{F}_v} \hat{\nu}_2(z).A_2(z)\, dz = 1$.

We define a Whittaker function W_3 in $\mathcal{W}_{\pi_v}$ by

$$(10.8)\qquad W_3(g) = \int_{\mathbf{F}_v} \nu_2(x). \left(\rho \begin{pmatrix} 1 & & & & & \\ & 1 & & -x & & \\ & & 1 & & x & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} W_2 \right) (g)\, dx.$$

Then

$$(10.9)\quad \int_{\mathbf{F}_v^{\times}} \int_{\mathbf{F}_v} W_3 \begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & & z & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} .|a|^{s-3}\, d^{\times}a\, dz = \int_{\mathbf{F}_v} \hat{\nu}_2(z).A_2(z)\, dz = 1.$$

• STEP 4: Let

$$A_3(u) = \int_{\mathbf{F}_v^{\times}} \int_{\mathbf{F}_v} W_3 \begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & u & z & 1 & & \\ & & -u & & 1 & \\ & & & & & 1 \end{pmatrix} .|a|^{s-3}\, dz\, d^{\times}a.$$

Then by (10.9) we have $A_3(0) = 1$. Suppose that ν_3: $\mathbf{F}_v \longrightarrow \mathbb{C}$ is a Schwartz function in $\mathcal{S}(\mathbf{F}_v)$ which is chosen such that its Fourier transform, defined by $\hat{\nu}_3(u) = \int_{\mathbf{F}_v} \nu_3(x).\psi_o(u.x)\,dx$, is supported on a sufficiently small neighborhood of 0, and $\int_{\mathbf{F}_v} \hat{\nu}_3(u).A_3(u)\,du = 1$.

We define a Whittaker function W_4 in $\mathcal{W}_{\pi_v}$ by

$$W_4(g) = \int_{\mathbf{F}_v} \nu_3(x).\left(\rho\begin{pmatrix}1&&&&&\\&1&&&x&\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}W_3\right)(g)\,dx. \tag{10.10}$$

Then

(10.11)

$$\int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v}\int_{\mathbf{F}_v} W_4\begin{pmatrix}a&&&&&\\&a&&&&\\&&a&&&\\&u&z&1&&\\&&-u&&1&\\&&&&&1\end{pmatrix}.|a|^{s-3}\,d^\times a\,du\,dz = \int_{\mathbf{F}_v} \hat{\nu}_3(u).A_3(u)\,du = 1.$$

• STEP 5: Recall the representation ρ which acts by right translation on the Whittaker space $\mathcal{W}_{\pi_v}$. Let W_5 be a Whittaker function in $\mathcal{W}_{\pi_v}$ such that $W_5 = \rho(\gamma)W_4$, hence we have $W_4 = \rho(\gamma^{-1})W_5$. Then (10.11) becomes

$$\int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v}\int_{\mathbf{F}_v} W_5\left(\begin{pmatrix}a&&&&&\\&a&&&&\\&&a&&&\\&u&z&1&&\\&&-u&&1&\\&&&&&1\end{pmatrix}.\gamma^{-1}\right).|a|^{s-3}\,d^\times a\,du\,dz = 1. \tag{10.12}$$

• STEP 6: Let us define a subgroup of $\mathbf{K}_v$,

$$\mathbf{K}_v(\wp^N) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathbf{K}_v, \text{where } c \equiv 0 \pmod{\wp^N} \text{ for some sufficiently large N}\right\}.$$

Then $(\mathbf{B}_{2,v} \cap \mathbf{K}_v) \subset \mathbf{K}_v(\wp^N)$, obviously. We choose a function $f^o_{s,v}$ which satisfies

$$f^o_{s,v}\left[\begin{pmatrix}y_1&x\\&y_2\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}\right] = \begin{cases}\delta^s_{\mathbf{B}_{2,v}}\begin{pmatrix}y_1&x\\&y_2\end{pmatrix} = |y_1/y_2|^s, \text{ if} \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathbf{K}_v(\wp^N),\\ 0, \text{ if} \begin{pmatrix}a&b\\c&d\end{pmatrix} \notin \mathbf{K}_v(\wp^N).\end{cases} \tag{10.13}$$

Then $f^o_{s,v}$ is a smooth function in the space $\mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta^s_{\mathbf{B}_{2,v}}$. Hence

$$f^o_{s,v}(\tilde{k}) = \begin{cases} 1 & \text{if} \quad k \in \mathbf{K}_v(\wp^N), \quad \text{or } \tilde{k} \in (\mathbf{B}_{2,v} \cap \mathbf{K}_v)\backslash \mathbf{K}_v(\wp^N), \\ 0 & \text{otherwise.} \end{cases} \tag{10.14}$$

Then recalling (10.5), for any Whittaker function W_h in $\mathcal{W}_{\pi_v}$, we have

$$\check{Z}_v(s, W_h, f^o_{s,v}) = \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v} \int\limits_{\mathbf{F}_v^\times} \int\limits_{\mathbf{F}_v^2} W_h\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix} .\gamma^{-1}.\mathbf{i}(\tilde{k})\right)$$
$$\times |a|^{s-3}.f^o_{s,v}(\tilde{k})dzdud^\times a\, d\tilde{k}$$
$$= \int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v^2}\int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v(\wp^N)} (\rho(\mathbf{i}(\tilde{k}))W_h)\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix} .\gamma^{-1}\right)$$
$$\times |a|^{s-3}\, dz\, du\, d^\times a\, d\tilde{k}. \tag{10.15}$$

• STEP 7: Using the Iwahori factorization, we have

$$(\mathbf{B}_{2,v} \cap \mathbf{K}_v)\backslash\mathbf{K}_v(\wp^N) \simeq \left\{ \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \in \mathbf{K}_v, \text{such that } c \equiv 0 \pmod{\wp^N} \right\}.$$

Then for any W_h in $\mathcal{W}_{\pi_v}$,

$$\int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v(\wp^N)} (\rho(\mathbf{i}(\tilde{k}))W_h)\, d\tilde{k} = \int\limits_{c\,\in\,\wp^N} \left(\rho\left[\mathbf{i}\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right] W_h\right) dc = q^{-N}.W_h,$$

if N is chosen sufficiently large, since W_h is locally constant. Take $W_h = q^N.W_5$, where W_5 was defined in step 5 above. Then

$$\int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v(\wp^N)} (\rho(\mathbf{i}(\tilde{k}))W_h)\, d\tilde{k} = W_5. \tag{10.16}$$

Thus the integral in (10.15) is equal to that in (10.12).

Let $W^o_v = W_h$. Then $\check{Z}_v(s, W^o_v, f^o_{s,v}) = 1$. It completes the proof of Propositions 10.4 and 10.2. ∎

CHAPTER II: THE POLES OF THE GLOBAL SPIN L-FUNCTION

11. Meromorphic continuation of the integral $Z_v(s, W_v, f_{s,v})$ at archimedean places

We will prove a proposition which will play the role of Proposition 10.2 for archimedean places.

Let $\mathbf{F}_v$ be an archimedean local field. The maximum compact subgroup $\mathbf{K}_v$ is $O(2,\mathbb{R})$ or $U(2,\mathbb{C})$, depending on $\mathbf{F}_v$ being a real field $\mathbb{R}$ or complex field $\mathbb{C}$, respectively.

PROPOSITION 11.1 (Meromorphic continuation of $Z_v(s, W_v, f_{s,v})$ at archimedean places): *The integral $Z_v(s, W_v, f_{s,v})$, which is defined in (10.5),*

$$Z_v(s,W_v,f_{s,v}) = \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v}\ \int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v^2} W_v\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.\gamma^{-1}.\mathrm{i}(\tilde{k})\right)$$
$$\times\, |a|^{s-3}.f_{s,v}(\tilde{k})\, dz\, du\, d^\times a\, d\tilde{k}, \tag{11.1}$$

converges for sufficiently large $\Re(s)$ and has a meromorphic continuation to all s.

To prove Proposition 11.1, we need to estimate the Whittaker functions W_v.

• PART A: ESTIMATES FOR THE WHITTAKER FUNCTIONS. We will consider some subgroups of $\mathbf{GSp}(6,\mathbf{F}_v)$. Let $\mathbf{K}_{6,v}$ be the maximum compact subgroup and $\mathbf{N}_v$ be the unipotent radical subgroup. We define the character $\Psi_{\mathbf{N}_v}$ on $\mathbf{N}_v$ as in Section 10.F.

Let $\delta_{\mathbf{B}_6}$ be the module of the Borel subgroup $\mathbf{B}_{6,v}$ and $\mathbf{D}_v$ be the subgroup consisting of all diagonal matrices of the form

$$\mathbf{a}(y_o,y_1,y_2,y_3) = \begin{pmatrix} y_oy_1y_2y_3 &&&&& \\ & y_oy_1y_2 &&&& \\ && y_oy_1 &&& \\ &&& y_o && \\ &&&& y_oy_2^{-1} & \\ &&&&& y_oy_2^{-1}y_3^{-1} \end{pmatrix}, \tag{11.2}$$

where all $y_i \in \mathbf{F}_v$. Let E_D be a finite set of characters $\mathcal{X}$ of $\mathbf{D}_v$. Each character $\mathcal{X}$ has the form

$$\mathcal{X}(\mathbf{a}(y_o,y_1,y_2,y_3)) = \prod_{i=0}^{3} \mathcal{X}_i(y_i)|y_i|^{n_i}, \tag{11.3}$$

where $\mathcal{X}_i$ is a character of module one and n_i is real.

Let E be the set of finite functions $\mathcal{Y}$ on $\mathbf{F}_v^4$. Then each function $\mathcal{Y}(\mathbf{a})$ is a finite linear combination of functions of the form

$$\mathcal{X}(\mathbf{a}(y_o, y_1, y_2, y_3)) \cdot \prod_{j=1}^{3} \log^{m_j} |y_j|, \tag{11.4}$$

where m_j's are positive integers. Then E is also a finite set.

PROPOSITION 11.2: *For every Whittaker function W_v in $\mathcal{W}_{\pi_v}$, there exist Schwartz functions Ψ_i's in $\mathcal{S}(\mathbf{F}_v^3 \times \mathbf{K}_{6,v})$ such that*

$$W_v(\mathbf{n}.\mathbf{a}.\mathbf{k}) = \Psi_{\mathbf{N}_v}(\mathbf{n}).\delta_{\mathbf{B}_6}^{1/2}(\mathbf{a}). \sum_{\mathcal{Y}_i \in E'} \Psi_i(y_1, y_2, y_3; \mathbf{k}).\mathcal{Y}_i(\mathbf{a}), \tag{11.5}$$

where $\mathbf{a} = \mathbf{a}(y_o, y_1, y_2, y_3) \in \mathbf{D}_v$, $\mathbf{k} \in \mathbf{K}_{6,v}$ and $\mathbf{n} \in \mathbf{N}_v$.

Proof: The proof is the same as in [S] which was inspired by those works in [J,S.1], [J,S.2] and [J,PS,S]. ∎

• PART B: PROOF OF PROPOSITION 11.1. By the Iwasawa decomposition, we have the following identity:

$$\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & u & z & 1 & & \\ & & -u & & 1 & \\ & & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & c_1 & * & * & 0 \\ & & 1 & c_2 & * & 0 \\ & & & 1 & c_1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

$$. \operatorname{diag}(a, a\delta, a\Delta, \Delta^{-1}, \delta^{-1}, 1).\mathbf{k} \tag{11.6}$$

$$= \mathbf{n}(a, u, z).\mathbf{a}(\Delta^{-1}, \Delta^2 a, \delta\Delta^{-1}, \delta^{-1}).\mathbf{k}, \tag{11.7}$$

for some $\mathbf{k} \in \mathbf{K}_{6,v}$,

$$\begin{gathered} c_1 = -zu/(1+u^2), \quad c_2 = za/[z^2 + (1+u^2)^2], \\ \delta = (1+u^2)^{-1/2} \quad \text{and} \quad \Delta = (1 + u^2 + [z^2/(1+u^2)])^{-1/2}. \end{gathered} \tag{11.8}$$

Let $\mathbf{k}_\gamma = \mathbf{k}.\gamma^{-1}.\mathbf{i}(\tilde{k})$; then $\mathbf{k}_\gamma$ is in $\mathbf{K}_{6,v}$ since both γ^{-1} and $\mathbf{i}(\tilde{k})$ are in $\mathbf{K}_{6,v}$. Applying the identity in (11.7) to Proposition 11.2, we have

$$W_v\left(\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & u & z & 1 & & \\ & & -u & & 1 & \\ & & & & & 1 \end{pmatrix}.\gamma^{-1}.\mathbf{i}(\tilde{k})\right)$$

$$= W_v\Big(\mathbf{n}(a,u,z).\mathbf{a}(\Delta^{-1},\Delta^2 a,\delta\Delta^{-1},\delta^{-1}).\mathbf{k}.\gamma^{-1}.\mathbf{i}(\tilde{k})\Big)$$

$$= \Psi_{\mathbf{N}_v}(\mathbf{n}(a,u,z)).\delta_{\mathbf{B}_6}^{1/2}(\mathbf{a}).\sum_{\mathcal{Y}_i\in E}\Big[\Psi_i(y_1,y_2,y_3;\mathbf{k}_\gamma)$$

$$\times\sum_{i=1}^{M}\Big(\mathcal{X}_o(y_o).|y_o|^{n_{o,i}}.\prod_{j=1}^{3}\Big[\mathcal{X}_j(y_j).|y_j|^{n_{j,i}}.\log^{m_{j,i}}|y_j|\Big]\Big)\Big] \tag{11.9}$$

(where $y_o = \Delta^{-1}$, $y_1 = \Delta^2 a$, $y_2 = \delta\Delta^{-1}$, $y_3 = \delta^{-1}$; and for the next step, we use a simple arithmetic formula: the power $\log^m|a.b|$ is a finite linear combination of products of the form $(\log^{m_1}|a|.\log^{m_2}|b|)$, with all positive integers $m_1+m_2=m$)

$$= \Psi_{\mathbf{N}_v}(\mathbf{n}(a,u,z)).\delta_{\mathbf{B}_6}^{1/2}(\mathbf{a}).\sum_{\mathcal{Y}_i\in E}\Big(\Psi_i(y_1,y_2,y_3;\mathbf{k}_\gamma).|a|^{\alpha_1}.|\Delta|^{\alpha_2}.|\delta|^{\alpha_3}.\prod_{l=o}^{3}\mathcal{X}_l(y_l)$$

$$\times\sum_{(\beta_1,\beta_2,\beta_3)} C_\beta.\log^{\beta_1}|a|.\log^{\beta_2}|\Delta|.\log^{\beta_3}|\delta|\Big), \tag{11.10}$$

where all β_i's are positive integers and both $\sum$'s are finite sums. Then the integral $Z_v(s,W_v,f_{s,v})$ in (11.1) is a finite linear combination of terms of the form

$$Y_j(s) = \int_{\mathbf{K}_{6,v}}\int_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v}\int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v^2}\Psi_{\mathbf{N}_v}(\mathbf{n}(a,u,z)).\delta_{\mathbf{B}_6}^{1/2}(\mathbf{a}).\Psi_j(y_1,y_2,y_3;\mathbf{k}_\gamma)$$

$$\times\prod_{l=o}^{3}\mathcal{X}_l(y_l).|a|^{s+\alpha_1}.|\Delta|^{\alpha_2}.|\delta|^{\alpha_3}.\log^{\beta_1}|a|.\log^{\beta_2}|\Delta|.\log^{\beta_3}|\delta|$$

$$\times f_{s,v}(\tilde{k})\,dz\,du\,d^\times a\,d\tilde{k}\,d\mathbf{k} \tag{11.11}$$

where, for $\mathbf{n}(a,u,z)$ expressed in (11.6)–(11.8), we have

$$\Psi_{\mathbf{N}_v}(\mathbf{n}(a,u,z)) = \psi_o\Big(\frac{-zu}{1+u^2}+\frac{za}{z^2+(1+u^2)^2}\Big), \tag{11.12}$$

where we defined $\psi_o(x) = e^{2\pi i\Re(x)}$ for archimedean places. Now $\delta_{\mathbf{B}_6}^{1/2}(\mathbf{a})$ is of the form $|a|^{\alpha_1'}.|\Delta|^{\alpha_2'}.|\delta|^{\alpha_3'}$, where $\Re(\alpha_1')\geq 0$. Thus it can be absorbed into this product:

$$\Xi_j(a,u,z;\mathbf{k}_\gamma) = \Psi_j(y_1,y_2,y_3;\mathbf{k}_\gamma).\prod_{l=o}^{3}\mathcal{X}_l(y_l).|a|^{\alpha_1}.|\Delta|^{\alpha_2}.|\delta|^{\alpha_3}.\log^{\beta_2}|\Delta|.\log^{\beta_3}|\delta|,$$

where $y_1 = a\Delta^2$, $y_2 = \delta\Delta^{-1}$ and $y_3 = \delta^{-1}$.

From the expressions of δ and Δ in (11.8), it is not difficult to observe that $\Xi_j(a, u, z; \mathbf{k}_\gamma)$ is still a Schwartz function. Therefore, we can simplify the expression of $Y_j(s)$ to

$$Y_j(s) = \int\limits_{\mathbf{K}_{6,v}} \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v} \int\limits_{\mathbf{F}_v^\times} \int\limits_{\mathbf{F}_v^2} \Big[\Psi_{\mathbf{N}_v}(\mathbf{n}(a, u, z)).\Xi_j(a, u, z; \mathbf{k}_\gamma) \tag{11.13}$$
$$\times |a|^s. \log^\beta |a| f_{s,v}(\tilde{k})\Big] dz\, du\, d^\times a\, d\tilde{k}\, d\mathbf{k}.$$

Therefore, in order to prove the convergence and meromorphic continuation of the integral $Z_v(s, W_v, f_{s,v}) = \sum_{j\in\mathcal{J}} Y_j(s)$, where $\mathcal{J}$ is some finite index set, it suffices to prove those properties of integrals of each summand $Y_j(s)$.

We recall that $\mathbf{k}_\gamma = \mathbf{k}.\gamma^{-1}.\mathbf{i}(\tilde{k})$ and $\mathbf{k}$ runs in some subset $\mathbf{K}'$ of the subgroup $\mathbf{K}_{6,v}$. For fixed a, u, z, s,

$$I_K = \int\limits_{\mathbf{K}'} \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v} \left|\Xi_j(a, u, z, \mathbf{k}_\gamma).f_{s,v}(\tilde{k})\right| d\tilde{k}\, d\mathbf{k}$$
$$\leq \int\limits_{\mathbf{K}_{6,v}} \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v} \left|\Xi_j[a, u, z; \mathbf{k}.\gamma^{-1}.\mathbf{i}(\tilde{k})].f_{s,v}(\tilde{k})\right| d\tilde{k}\, d\mathbf{k}.$$

The latter integrals converge because they are integrals of smooth functions on compact domains. Therefore, I_K also converges to some function $\Upsilon_j(a, u, z; s)$ which is still a Schwartz function on $\mathbf{F}_v^3$ of variables a, u, z and a smooth function of variable s. Then, we rewrite

$$Y_j(s) = \int\limits_{\mathbf{F}_v} \int\limits_{\mathbf{F}_v} \int\limits_{\mathbf{F}_v^\times} \Psi_{\mathbf{N}_v}(\mathbf{n}(a, u, z)).\Upsilon_j(a, u, z; s).|a|^s. \log^\beta |a|\, dz\, du\, d^\times a, \tag{11.14}$$

where β is a positive integer.

Suppose for definiteness that $\mathbf{F}_v = \mathbb{R}$. Then it suffices to consider the integral of the following form:

(11.5)

$$Y_j(s) = \int_0^\infty \int_0^\infty \int_0^\infty \Psi_{\mathbf{N}_v}(\mathbf{n}(a, u, z)).\Upsilon_j(a, u, z; s).a^s. \log^\beta(a)\, dz\, du\, d^\times a$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \psi_o\Big(\frac{-zu}{1+u^2} + \frac{za}{z^2 + (1+u^2)^2}\Big).\Upsilon_j(a, u, z; s).a^{s-1}.\log^\beta(a)\, dz\, du\, da.$$

(When $\mathbf{F}_v = \mathbb{C}$, we can observe easily that all the extra terms and integrals can be absorbed in the product $(\Upsilon_j(a,u,z;s).a^{s-1})$; thus there is no loss of generality.)

Now we partition $Y_j(s)$ into two integrals: $Y_j(s) = Y_1(s) + Y_2(s)$, where

$$Y_1(s) = \int_0^1\int_0^\infty\int_0^\infty \psi_o\Big(\frac{-zu}{1+u^2}+\frac{za}{z^2+(1+u^2)^2}\Big).\Upsilon_j(a,u,z;s).a^{s-1}.\log^\beta(a)\,dz\,du\,da,$$

and

$$Y_2(s) = \int_1^\infty\int_0^\infty\int_0^\infty \psi_o\Big(\frac{-zu}{1+u^2}+\frac{za}{z^2+(1+u^2)^2}\Big).\Upsilon_j(a,u,z;s).a^{s-1}.\log^\beta(a)\,dz\,du\,da.$$

Then

$$|Y_2(s)| = \int_1^\infty\int_0^\infty\int_0^\infty \Big|\Upsilon_j(a,u,z;s).a^{s-1}.\log^\beta(a)\Big|.\,dz\,du\,da$$

converges for all s because the Schwartz function $\Upsilon_j(a,u,z;s)$ approaches 0 faster than any polynomial of variable a, and the integrals w.r.t. variables u and z are convergent. We also have

$$\begin{aligned}|Y_1(s)| &= \int_0^1\int_0^\infty\int_0^\infty \Big|\Upsilon_j(a,u,z;s).a^{s-1}.\log^\beta(a)\Big|\,dz\,du\,da\\ &= \int_0^1 \Omega_j(a).|a^{s-1}|.|\log^\beta(a)|\,da,\end{aligned}$$

where $\Omega_j(a)$ is a smooth function of a. This integral converges when $\Re(s)$ is sufficiently large. Therefore, $Y_j(s)$ is convergent for $\Re(s)$ sufficiently large.

Now we will prove the existence of a meromorphic continuation of $Y_1(s)$ which will imply the same for $Y_j(s)$. The integral

$$\int_0^\infty\int_0^\infty \psi_o\Big(\frac{-zu}{1+u^2}+\frac{za}{z^2+(1+u^2)^2}\Big).\Upsilon_j(a,u,z;s)\,dz\,du \tag{11.16}$$

converges to a smooth function $G(a)$. The Taylor expansion near $a=0$ gives us (11.17)

$$G(a) = \sum_{k=0}^{N-1} C_k.a^k + R(a), \qquad \text{where } R(a) = O(a^N), \qquad \text{for some large } N.$$

Then

$$Y_1(s) = \sum_{k=0}^{N-1} C_k. \int_0^1 a^{s+k-1}. \log^\beta(a)\, da + C_N. \int_0^1 a^{s+N-1}. \log^\beta(a)\, da. \tag{11.18}$$

By Lemma 11.5 proved below, each term in the right hand side of (11.18) has a meromorphic continuation to all s, and so do $Y_1(s)$ and $Y_j(s)$. This completes the proof of Proposition 11.1. ∎

Lemma 11.5: *For any integer $\beta > 0$, the integral $Q(s,\beta) = \int_0^1 a^{s-1}. \log^\beta(a)\, da$ has a meromorphic continuation to the whole complex plane.*

Proof: Integration by parts then gives us

$$Q(s,\beta) = \frac{a^s}{s} \cdot \log^\beta(a)\Big|_{a\to 0}^{a=1} - \frac{\beta}{s} \cdot Q(s,\beta-1). \tag{11.19}$$

We have

$$R(s,\beta) = \frac{a^s}{s} \cdot \log^\beta(a)\Big|_{a\to 0}^{a=1} = \lim_{a\to 0} \frac{a^s}{s} \cdot \log^\beta(a) = 0, \quad \text{when } \Re(s) \geq 1.$$

Therefore, the continuation of $Q(s,\beta)$ will depend on that of $Q(s,\beta-1)$.

This recursive relationship reduces to proving meromorphic continuation of the integral $Q(s,0) = \int_0^1 a^{s-1}\, da$, which is already known. ∎

12. Non-vanishing of the integral $Z_v(s, W_v, f_{s,v})$ at archimedean places

Now we will prove a proposition which will play the role of Proposition 10.4 for archimedean places. Let $\mathbf{F}_v$ be an archimedean local field and $\mathbf{K}_v = O(2,\mathbb{R})$ or $U(2,\mathbb{C})$ depending on $\mathbf{F}_v$ being real $\mathbb{R}$ or complex $\mathbb{C}$, respectively.

Proposition 12.1: *Let v be an archimedean local place. For any s_o fixed, there exist a Whittaker function $W_v^o \in \mathcal{W}_{\pi_v}$ and a $\mathbf{K}_v$-finite function $f_{s,v}^o \in \mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^s$ such that the meromorphic continuation of the integral*

$$Z_v(s, W_v^o, f_{s,v}^o) = \int_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v} \int_{\mathbf{F}_v^\times} \int_{\mathbf{F}_v^2} W_v^o\left(\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix} .\gamma^{-1}.\mathbf{i}(\tilde{k})\right)$$

$$\times\, |a|^{s-3}. f_{s,v}^o(\tilde{k})\, dz\, du\, d^\times a\, d\tilde{k} \tag{12.1}$$

does not vanish at $s = s_o$.

Proof:

- STEP 1: Let $\mathcal{S}(\mathbf{F}_v)$ be the set of all Schwartz functions $\nu\colon \mathbf{F}_v \longrightarrow \mathbb{C}$.

There is some Whittaker function W_1 in the space $\mathcal{W}_{\pi_v}$ such that $W_1(1) \neq 0$, and then it can be normalized to $W_1(1) = 1$.

- STEP 2: Let $A_1(a) = W_1(\mathrm{diag}(a,a,a,1,1,1))$. Then

$$A_1(1) = 1. \tag{12.2}$$

For any Schwartz function ν_1 in $\mathcal{S}(\mathbf{F}_v)$, its Fourier transform, defined by $\hat{\nu}_1(a) = \int_{\mathbf{F}_v} \nu_1(x).\psi_o(a.x)\,dx$, is also a Schwartz function in $\mathcal{S}(\mathbf{F}_v)$. We define a Whittaker function W_2 in $\mathcal{W}_{\pi_v}$ by

$$W_2(g) = \int_{\mathbf{F}_v} \nu_1(x).(\rho\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&x&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}W_1)(g)\,dx. \tag{12.3}$$

First, we assume that $\Re(s_o)$ is sufficiently large. Then by the same manipulation in step 2, Section 12, we can work on the following convergent integral:

$$\int_{\mathbf{F}_v^\times} W_2(\mathrm{diag}(a,a,a,1,1,1)).|a|^{s_o-3}\,d^\times a = \int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v} \nu_1(x).\psi_o(a.x).A_1(a).|a|^{s_o-3}\,d^\times a$$

$$= \int_{\mathbf{F}_v^\times} \hat{\nu}_1(a).A_1(a).|a|^{s_o-3}\,d^\times a. \tag{12.4}$$

CLAIM: *We can choose some Schwartz function ν_1 such that the integral* (12.4) *does not vanish.*

Indeed, if not so, then the integral (12.4) has to be 0 for all Schwartz functions ν_1 in $\mathcal{S}(\mathbf{F}_v)$. It forces $A_1(a).|a|^{s_o-3} = 0$ for all a. When $a = 1$, this gives $A_1(1) = W_1(1) = 0$ which contradicts the result (12.2). Thus, there exists some function ν_1 such that W_2 defined in (12.3) satisfies

$$\int_{\mathbf{F}_v^\times} W_2(\mathrm{diag}(a,a,a,1,1,1)).|a|^{s_o-3}\,d^\times a \neq 0. \tag{12.5}$$

Moreover, we can modify the function ν_1 such that $\hat{\nu}_1$ approaches 0 rapidly when $a \to 0$. Then the integrals in (12.4) are convergent for all s_o. Therefore, the result (12.5) is also true for all s_o.

• STEP 3: Let

$$A_2(z; s_o) = \int_{\mathbf{F}_v^\times} W_2 \begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & & z & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} .|a|^{s_o - 3}\, d^\times a.$$

Then by (12.5) we have

$$A_2(0; s_o) \neq 0, \quad \text{for all } s_o. \tag{12.6}$$

We need the following lemma.

LEMMA 12.2: *The function $A_2(z; s_o)$ has a meromorphic continuation in variable s_o and, for a fixed s_o, it is an analytic function of variable z.*

Proof: We can estimate the function Whittaker W_2 by the same steps as in (11.6)–(11.13), with $u = 0$. Then

$$A_2(z; s_o) = \sum_{j \in \mathcal{J}} \int_{\mathbf{F}_v^\times} \psi_o\Big(\frac{za}{1+z^2}\Big).\Xi_j(a, z; \mathbf{k}_\gamma).|a|^{s_0 - 3}. \log^\beta |a|\, d^\times a,$$

where $\Xi_j(a, z; \mathbf{k}_\gamma)$ is a Schwartz function, and $\mathcal{J}$ is some finite index set. Therefore, A_2 is an analytic function of variable z; and the proof of meromorphic continuation will follow the same arguments as in the proof of Proposition 11.1. ∎

Let ν_2: $\mathbf{F}_v \longrightarrow \mathbb{C}$ be a Schwartz function in $\mathcal{S}(\mathbf{F}_v)$. We define a Whittaker function W_3 in $\mathcal{W}_{\pi_v}$ by

$$W_3(g) = \int_{\mathbf{F}_v} \nu_2(x).(\rho \begin{pmatrix} 1 & & & & & \\ & 1 & & -x & & \\ & & 1 & & x & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} W_2)(g)\, dx. \tag{12.7}$$

Then the same manipulation in step 3, Section 12, gives us

$$\int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v} W_3\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ && z & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}.|a|^{s_o-3}\, d^\times a\, dz = \int_{\mathbf{F}_v} \hat{\nu}_2(z).A_2(z;s_o)\, dz, \tag{12.8}$$

for $\Re(s_o)$ sufficiently large and $\hat{\nu}_2$ is the Fourier transform of ν_2, defined by

$$\hat{\nu}_2(z) = \int_{\mathbf{F}_v} \nu_2(x).\psi_o(z.x)\, dx.$$

CLAIM: *We can choose some Schwartz function ν_2 such that the integral* (12.8) *does not vanish.*

Indeed, if not so, the integral (12.8) has to be 0 for all Schwartz functions ν_2 in $\mathcal{S}(\mathbf{F}_v)$. It forces $A_2(z;s_o) = 0$ for all z. When $z = 0$, this gives $A_2(0;s_o) = 0$ which contradicts the result (12.6). Thus, there exists some function ν_2 such that W_3 defined in (12.7) satisfies

$$\int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v} W_3\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ && z & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}.|a|^{s_o-3}\, d^\times a\, dz \neq 0. \tag{12.9}$$

Moreover, we can modify the function ν_2 such that $\hat{\nu}_2$ approaches 0 rapidly when $z \to \infty$. Then the integrals in (12.8) are convergent for all s_o. Therefore, the result in (12.9) is also true for all s_o.

• STEP 4: Let

$$A_3(u;s_o) = \int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v} W_3\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.|a|^{s_o-3}\, dz\, d^\times a.$$

Then by (12.9) we have

$$A_3(0;s_o) \neq 0, \qquad \text{for all } s_o. \tag{12.10}$$

LEMMA 12.3: *The function $A_3(u;s_o)$ has a meromorphic continuation in variable s_o and, for a fixed s_o, it is an analytic function of variable u.*

Proof: We can estimate the function Whittaker W_3 as in (11.6)–(11.13). Then

$$A_3(u;s_o) = \sum_{j\in\mathcal{J}} \int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v} \psi_o\Big(\frac{-zu}{1+u^2} + \frac{za}{z^2+(1+u^2)^2}\Big).\Xi_j(a,u,z;\mathbf{k}_\gamma) \times |a|^{s_o-3}.\log^\beta|a|\,dz\,d^\times a,$$

where $\Xi_j(a,u,z;\mathbf{k}_\gamma)$ is a Schwartz function and $\mathcal{J}$ is some finite index set. Then A_3 is an analytic function of variable u. The proof of meromorphic continuation will follow the same arguments as those in the proof of Proposition 11.1. ∎

Let $\nu_3\colon \mathbf{F}_v \longrightarrow \mathbb{C}$ be a Schwartz function in $\mathcal{S}(\mathbf{F}_v)$. We define a Whittaker function W_4 in $\mathcal{W}_{\pi_v}$ by

$$W_4(g) = \int_{\mathbf{F}_v} \nu_3(x).(\rho\begin{pmatrix}1&&&&&\\&1&&x&&\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}W_3)(g)\,dx. \tag{12.11}$$

Then the same manipulation in step 4, Section 12, gives us

(12.12)

$$\int_{\mathbf{F}_v^\times}\int_{\mathbf{F}_v}\int_{\mathbf{F}_v} W_4\begin{pmatrix}a&&&&&\\&a&&&&\\&&a&&&\\&u&z&1&&\\&&-u&&1&\\&&&&&1\end{pmatrix}.|a|^{s_o-3}\,d^\times a\,du\,dz = \int_{\mathbf{F}_v}\hat{\nu}_3(u).A_3(u;s_o)\,du,$$

for $\Re(s_o)$ sufficiently large and $\hat{\nu}_3$ is the Fourier transform of ν_3, defined by

$$\hat{\nu}_3(u) = \int_{\mathbf{F}_v} \nu_3(x).\psi_o(u.x)\,dx.$$

CLAIM: *We can choose some Schwartz function ν_3 such that the integral* (12.12) *does not vanish.*

Indeed, if not so, the integral (12.12) has to be 0 for all Schwartz functions ν_3 in $\mathcal{S}(\mathbf{F}_v)$. It forces $A_3(u;s_o)=0$ for all u. When $u=0$, this gives $A_3(0;s_o)=0$

which contradicts the result (12.10). Thus, there exists some function ν_3 such that W_4 defined in (12.11) satisfies

$$\int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v}\int\limits_{\mathbf{F}_v} W_4\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.|a|^{s_o-3}\, d^\times a\, du\, dz \neq 0. \tag{12.13}$$

Moreover, we can modify the function ν_3 such that $\hat{\nu}_3$ approaches 0 rapidly when $u \to \infty$. Then the integrals in (12.12) are convergent for all s_o. Therefore, the result in (12.13) is also true for all s_o.

• STEP 5: Let W_5 be a Whittaker function in $\mathcal{W}_{\pi_v}$ such that $W_5 = \rho(\gamma)W_4$, hence $W_4 = \rho(\gamma^{-1})W_5$. Then (12.13) becomes

$$\int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v}\int\limits_{\mathbf{F}_v} W_5\left\{\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.\gamma^{-1}\right\}.|a|^{s_o-3}\, d^\times a\, du\, dz \neq 0, \tag{12.14}$$

for all s_o in the complex plane.

• STEP 6: We recall the definition of $Z_v(s_o, W_v, f_{s_o,v})$ in (12.1):

$$Z_v(s_o, W_v, f_{s_o,v}) = \int\limits_{(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v}\int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v^2} W_v\left\{\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.\gamma^{-1}.\mathbf{i}(\tilde{k})\right\}$$

$$\times\, |a|^{s_o-3}.f_{s_o,v}(\tilde{k})\, dz\, du\, d^\times a\, d\tilde{k}. \tag{12.15}$$

Now two archimedean places will be considered separately.

Real place: $\mathbf{K}_v = O(2,\mathbb{R})$. Then

$$\mathbf{T}_v = (\mathbf{B}_{2,v}\cap\mathbf{K}_v) = \left\{\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \middle| a_1 = \pm 1, a_2 = \pm 1\right\},$$

and $\mathbf{Z} = \{\pm I\}$ is the center of the subgroup $\mathrm{SO}(2,\mathbb{R})$. ($I$ is the 2×2 identity matrix.) Thus

$$(\mathbf{B}_{2,v}\cap\mathbf{K}_v)\backslash\mathbf{K}_v \simeq \mathbf{T}_v\backslash O(2,\mathbb{R}) \simeq \mathbf{Z}\backslash\mathrm{SO}(2,\mathbb{R}) \simeq \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| 0 \le \theta \le \pi\right\}.$$

Let

$$z(\theta) = \int\limits_{\mathbf{F}_v^\times}\int\limits_{\mathbf{F}_v^2} W_5\left[\begin{pmatrix} a &&&&& \\ & a &&&& \\ && a &&& \\ & u & z & 1 && \\ && -u && 1 & \\ &&&&& 1 \end{pmatrix}.\gamma^{-1}.\mathbf{i}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right]$$

$$\times\, |a|^{s_o-3}\, dz\, du\, d^\times a. \tag{12.16}$$

Hence by (12.14),

$$z(0) \neq 0. \tag{12.17}$$

The representation ρ, which acts by right translation on the space $\mathcal{W}_{\pi_v}$, satisfies

$$\rho\left(\mathbf{i}\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right) W_5 = W_5.$$

Therefore

$$z(\theta+\pi) = z(\theta). \tag{12.18}$$

Thus in the Fourier expansion $z(\theta) = \sum_{-\infty}^{\infty} a_n.e^{in\theta}$, we have $a_n = 0$ if n is odd. Hence there exists some even N such that

$$0 \neq a_N = \int_0^\pi e^{-iN\theta} z(\theta)\, d\theta. \tag{12.19}$$

We choose the function $f_{s_o,v} = f_{s_o,v}^o$ which satisfies

$$f_{s_o,v}^o\left[\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix}\cdot\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right] = e^{-iN\theta}.\delta_{\mathbf{B}_{2,v}}^{s_o}\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix}, \tag{12.20}$$

where N is the even integer in (12.19). Then $f_{s_o,v}^o$ is a well-defined smooth function in the space $\mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}} \delta_{\mathbf{B}_{2,v}}^{s_o}$. Let $W_v = W_5$. Then, from (12.15) and (12.16), we have

$$Z_v(s_o, W_5, f_{s_o,v}^o) = \int\limits_0^\pi f_{s_o,v}^o\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} z(\theta)\, d\theta$$

$$= \int\limits_0^\pi e^{-iN\theta}.z(\theta)\, d\theta = \pi.a_N \neq 0. \tag{12.21}$$

Complex place:

$$\mathbf{K}_v = U(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \Bigm| |a|^2 + |b|^2 = 1; |a_1| = |a_2| = 1 \right\}.$$

Let

$$\mathbf{T}_v = (\mathbf{B}_{2,v} \cap \mathbf{K}_v) = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \Bigm| |a_1| = |a_2| = 1 \right\}.$$

Then $\mathbf{T}_v$ is the maximum torus subgroup and

$$\mathbf{T}_v \backslash \mathbf{K}_v \simeq \mathrm{SU}(2,\mathbb{C}) \simeq \left\{ \tilde{k} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \Bigm| |a|^2 + |b|^2 = 1 \right\}.$$

The center of $\mathrm{SU}(2,\mathbb{C})$ is just $\mathbf{Z} = \{\pm I\}$. We define a function z on $\mathrm{SU}(2,\mathbb{C})$:
(12.22)

$$z(\tilde{k}) = \int_{\mathbf{F}_v^\times} \int_{\mathbf{F}_v^2} W_5 \left(\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & u & z & 1 & & \\ & & -u & & 1 & \\ & & & & & 1 \end{pmatrix} .\gamma^{-1}.\mathrm{i}(\tilde{k}) \right) \times |a|^{s_o - 3} \, dz \, du \, d^\times a.$$

We have $z(1) \neq 0$. Similar to (12.18), we have $z(-\tilde{k}) = z(\tilde{k})$. Thus z is a $\mathbf{Z}$-invariant function in the space $L^2(\mathrm{SU}(2,\mathbb{C}))$.

Let (α_i, V_{α_i}) be a set of irreducible, finite-dimensional (hence unitary, by a properly chosen Hermitian inner product in V_{α_i}) representations of $\mathrm{SU}(2,\mathbb{C})$ such that the center $\mathbf{Z}$ of $\mathrm{SU}(2,\mathbb{C})$ acts trivially.

We can choose an orthonormal basis $v_1^i, \ldots, v_{m_i}^i$ of V_{α_i} and consider m_i^2 matrix coefficients of the form of the Hermitian inner product $\langle \alpha_i(\tilde{k}) v_j^i, v_l^i \rangle$ in the space V_{α_i}, for all $\tilde{k} \in \mathrm{SU}(2,\mathbb{C})$.

By the Peter–Weyl theorem, the union of these matrix coefficients over all α_i's (in fact, only over the equivalent classes of irreducible unitary representations) forms a complete orthonormal basis for the space $L^2(\mathrm{SU}(2,\mathbb{C}))$.

We can take a matrix coefficient t which is not orthogonal to z (because, if otherwise, z must be identical to 0) and also satisfies the same condition as on function z above: $t(-\tilde{k}) = t(\tilde{k})$. Then

$$\int_{\mathrm{SU}(2,\mathbb{C})} z(\tilde{k}).t(\tilde{k}) \, d\tilde{k} \neq 0. \tag{12.23}$$

We choose the function $f_{s_o,v} = f^o_{s_o,v}$ which satisfies

$$f^o_{s_o,v}\left[\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix}.\tilde{k}\right] = t(\tilde{k}).\delta^{s_o}_{\mathbf{B}_{2,v}}\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix}. \tag{12.24}$$

Then $f^o_{s_o,v}$ is a well-defined smooth function in the space $\mathrm{ind}_{\mathbf{B}_{2,v}}^{\mathbf{GL}_{2,v}}\delta^{s_o}_{\mathbf{B}_{2,v}}$. Let us choose $W_v = W_5$. Recalling (12.15) and (12.22), we have

$$Z_v(s_o, W_5, f^o_{s_o,v}) = \int_{\mathrm{SU}(2,\mathbb{C})} f^o_{s_o,v}(\tilde{k})z(\tilde{k})\,d\tilde{k} = \int_{\mathrm{SU}(2,\mathbb{C})} t(\tilde{k})z(\tilde{k})\,d\tilde{k} \neq 0. \tag{12.25}$$

This completes the proof of Proposition 12.1. ∎

13. The location of the poles of $L_S(s,\pi,\mathbf{spin})$

We recall the global spin L-function introduced in Section 0:

$$L_S(s,\pi,\mathrm{spin}) = \prod_{v\notin S} L_v(s,\pi_v,\mathrm{spin}). \tag{13.1}$$

THEOREM 13.1: *Let π be an irreducible, smooth and generic automorphic cuspidal representation of the symplectic groups $\mathbf{GSp}(6,\mathbf{F})$. The possible poles of the global spin L-function $L_S(s,\pi,spin)$ are only simple poles at $s = 0$ and $s = 1$.*

Proof: For all $v \notin S$, by the theorem 1 in [B,G], we have

$$Z_v(s, W_v, f_{s,v}) = L_v(s,\pi_v,\mathrm{spin}). \tag{13.2}$$

Then

$$\begin{aligned} Z(s,W,f_s) &= \prod_{\text{all } v} Z_v(s,W_v,f_{s,v}) = \prod_{v\notin S} L_v(s,\pi_v,\mathrm{spin}).\prod_{v\in S} Z_v(s,W_v,f_{s,v}) \\ &= L_S(s,\pi,\mathrm{spin}).\prod_{v\in S} Z_v(s,W_v,f_{s,v}), \end{aligned} \tag{13.3}$$

where $f_s = \prod_{\mathrm{all}\,v}(f_{s,v})$, and the Whittaker function $W = \prod_{\mathrm{all}\,v} W_v$.

For each non-archimedean local place $v \in S$, Proposition 10.4 allows us to choose local data W^o_v and $f^o_{s,v}$ such that $\check{Z}_v(s, W^o_v, f^o_{s,v}) = 1$ for all s. Then

$$Z_v(s, W^o_v, f^o_{s,v}) = \zeta_{\mathbf{F}}(2s) = (1 - \mathbf{N}_v^{-2s})^{-1} \neq 0.$$

Similarly, for each archimedean local place $v \in S$, Proposition 12.1 gives us the choice of local data W_v^o and $f_{s,v}^o$ such that, for any s, $Z_v(s, W_v^o, f_{s,v}^o) \neq 0$.

Thus at each s, we can choose the local data W_v^o's and $f_{s,v}^o$'s, for all $v \in S$, such that the finite product $\prod_{v \in S} Z_v(s, W_v^o, f_{s,v}^o)$ does not vanish.

Then, by (13.3), the poles of $L_S(s, \pi, \text{spin})$ are exactly the poles of the integral $Z(s, W^o, f_s^o)$, where

$$f_s^o = \prod_{v \notin S} f_{s,v} \cdot \prod_{v \in S} f_{s,v}^o \quad \text{and} \quad W^o = \prod_{v \notin S} W_v \cdot \prod_{v \in S} W_v^o.$$

Again, by Theorem 1 in [B,G], the possible poles of $Z(s, W^o, f_s^o)$ are only simple poles at $s = 0$ and $s = 1$. Therefore, they are also the possible poles of the global spin L-function $L_S(s, \pi, \text{spin})$. ∎

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